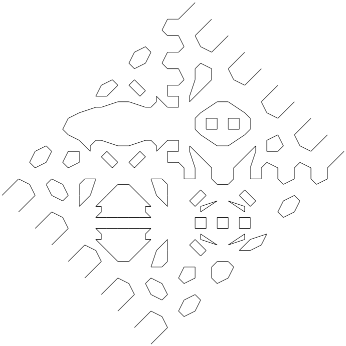
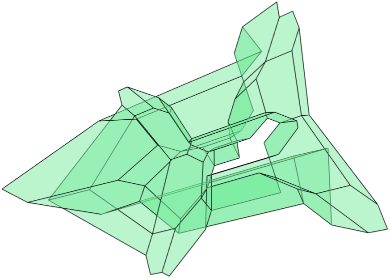
# Patchworking with tropical hypersurfaces in Polymake





Paul Vater
MPI MiS Leipzig

# Hilbert's 16th problem

For given $d$ find the possible isotopy types of real algebraic curves of degree $d$ (i.e., possible topology + relative position of connected components)!

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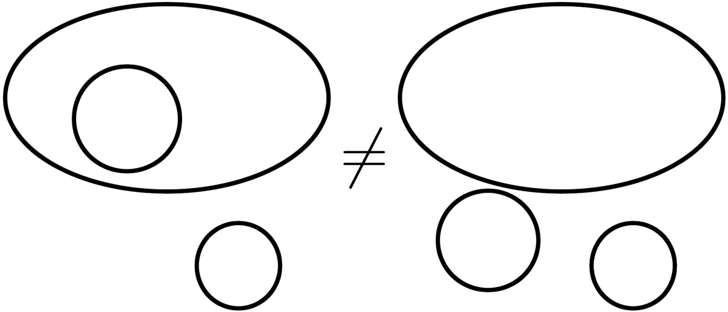
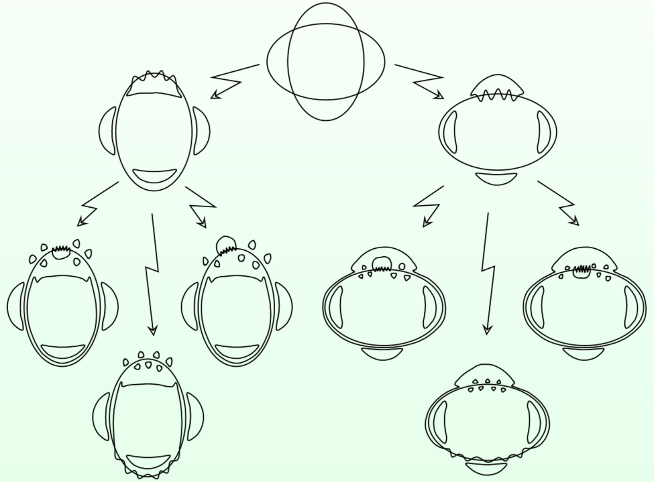


# Original methods (Harnack, Hilbert, . . . )

Find types of curves by "perturbing" curves of smaller degree:



picture by Oleg Viro

# Viro's patchworking method

**The Algorithm**

Input: Triangulation $\tau$ of $d \cdot \Delta_2 \cap \mathbb{Z}^n$, sign distribution $s : \mathrm{vert}(\tau) \to \mathbb{Z}_2$

Output: Polygonal curve $C$

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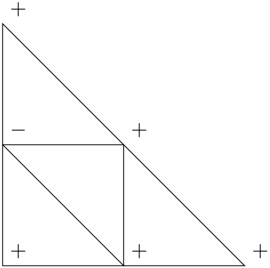

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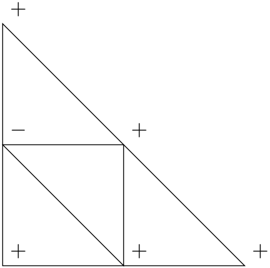


## Thm. (Viro)

$C$ represents the isotopy type of a certain real algebraic curve of degree $d$.

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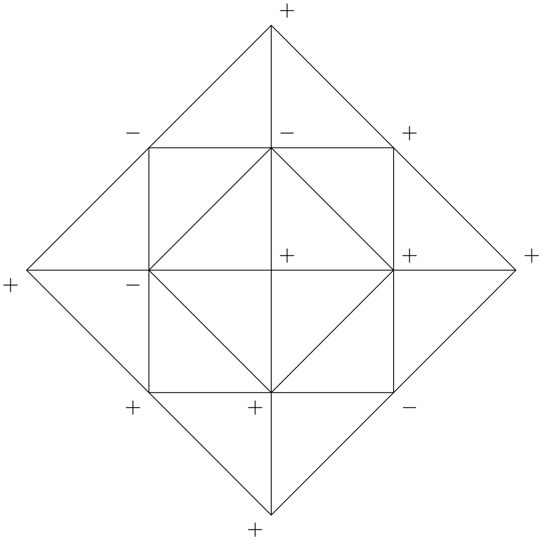

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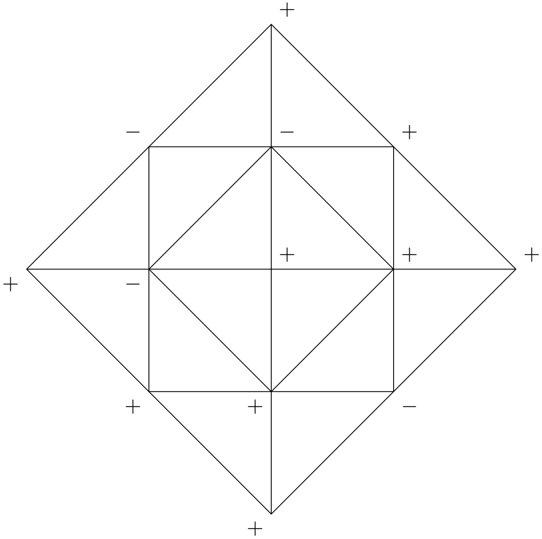

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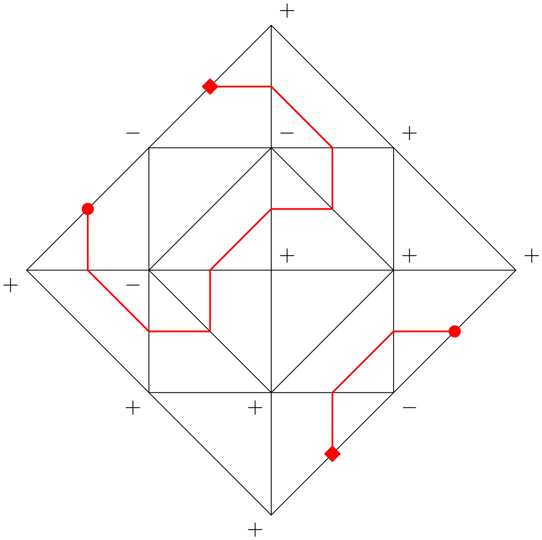

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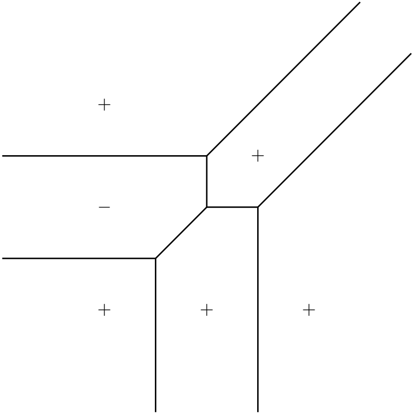

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- *real phase structure on* $\mathcal{T}(f)$:

$$\theta \subset \mathrm{facets}(\mathcal{T}(f)) \times \mathbb{Z}_2^n \quad \text{such that}$$
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## Tropical version of Viro's theorem

Realizing $\mathbb{R}\mathcal{T}_\theta(f)$ (in $\mathbb{R}^n$ or $\mathbb{R}P^n$) yields the isotopy type of a real algebraic hypersurface.

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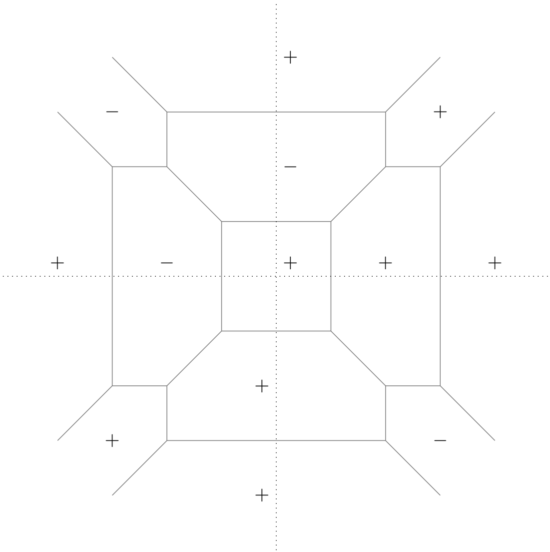


Figure: The symmetrized sign distribution.

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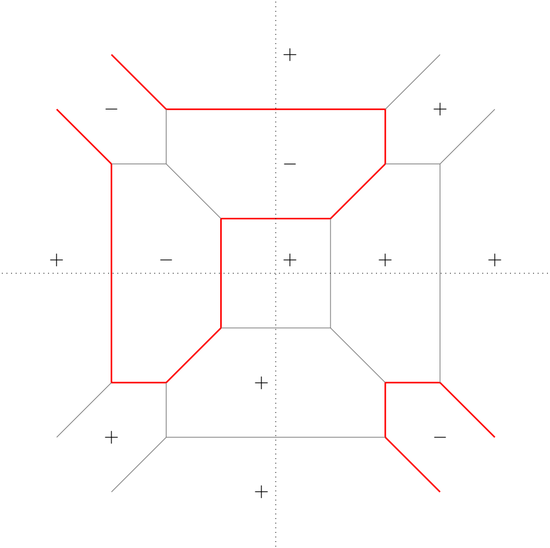


Figure: A realization of $\mathbb{R}\mathcal{T}_\theta(f)$.

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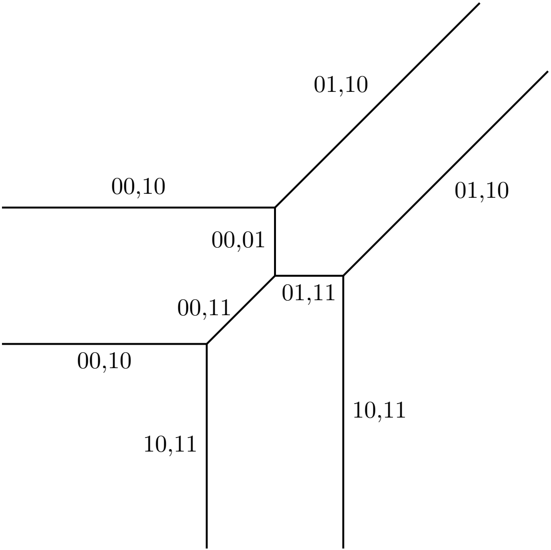


Figure: The real phase structure $\theta$ of our example.

# Polymake. . .

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$$\begin{aligned} C_q(\mathbb{R}\mathcal{T}_s(f);\mathbb{Z}_2) &= \bigoplus_{\sigma'=\sigma\times\{z\}<\mathbb{R}\mathcal{T}_s(f),\dim\sigma=q} \mathbb{Z}_2^{\{\sigma'\}} \\ &= \bigoplus_{\sigma<\mathcal{T}_s(f),\dim\sigma=q} \underbrace{\left(\bigoplus_{z\in\theta_s(\sigma)} \mathbb{Z}_2^{\{\sigma\times\{z\}\}}\right)}_{\mathcal{S}_{\theta_s}(\sigma)} \\ &= C_q(\mathcal{T}(f);\mathcal{S}_{\theta_s}) \end{aligned}$$

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$$\partial'(\sigma\times\{z\}) = \partial(\sigma)\times\{z\}$$

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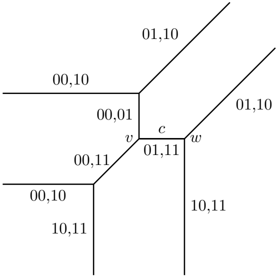
Example:



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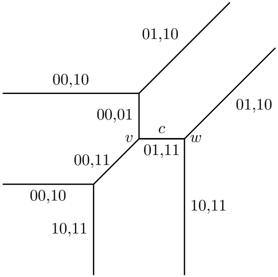


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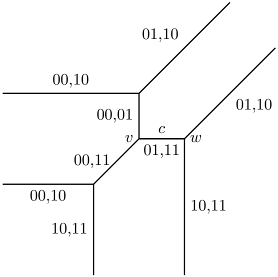


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$$\partial = \begin{array}{c} \\ \vdots \\ c \times 01 \\ c \times 11 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccc} \cdots & v \times 00 & v \times 01 & v \times 11 & \cdots & w \times 01 & \cdots \end{array} \\ \begin{pmatrix} & \vdots & \vdots & \vdots & & \vdots & \\ \cdots & 0 & 1 & 0 & \cdots & 1 & \cdots \\ & 0 & 0 & 1 & & 0 & \\ & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix} \end{array}$$

Polymake (WiP). . .