From dependent type theory to higher algebraic structures

Théories à types dépendants et algèbre de dimension supérieure

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A very informal introduction to this thesis

A 20$^{\text{th}}$-century relaying of the foundations of maths

There is a paradigm shift taking place in modern mathematics.

From sets to spaces

The basic objects of mathematics are *not* **sets**, but **spaces**.

From set theory to "space theory"

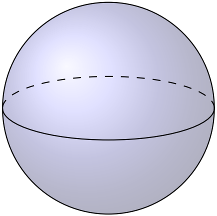
Mathematics has so far been founded on set theory.

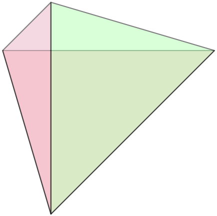
If spaces are the basic objects, we need a formal **theory of spaces** in which to do mathematics.

Every set is a space but not *vice versa*

A set is a collection of distinct *points* . . . but a space has points, and:

- its points can be joined by *lines*,
- loops can be covered by *surfaces*,
- holes can be filled by *volumes* . . .

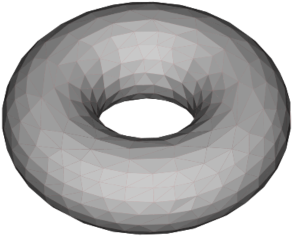




. . . and so on in all (finite) dimensions.

Spaces are the same up to deformation

By squishing and stretching a coffee cup (with a handle)...

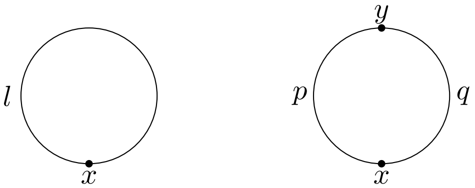


... we get a doughnut, so they're the same (or **equivalent**) spaces.

The basic structure of a space

A space is *presented* by its points, lines, surfaces, volumes etc. in all dimensions. We call these its **paths**.

The same space can have different presentations:



Both these equivalent presentations are “the circle”.

$$x \mapsto x \quad , \quad l \mapsto (q \cdot p)$$

$$x \leftarrow\!\dashv x \;,\; x \leftarrow\!\dashv y \quad , \quad l \leftarrow\!\dashv p \;,\; \mathsf{refl}_x \leftarrow\!\dashv q$$

Path spaces

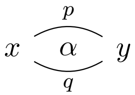
It makes sense to talk of "paths" of all dimension.

Id-ception

For any points x and y of a space S, there is a space $\mathrm{Id}_S(x, y)$

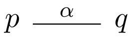
- whose points are lines $p\colon x \sim y$ in S between x and y,
- lines in $\mathrm{Id}_S(x, y)$ are surfaces $\alpha\colon p \sim q$ in S,
- surfaces in $\mathrm{Id}_S(x, y)$ are volumes in S . . . and so on (all dimensions shift by 1).

So



in S,

looks like

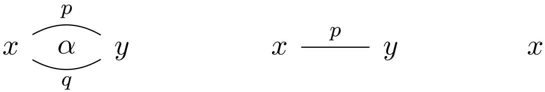


in $\mathrm{Id}_S(x, y)$.

Paths and equality

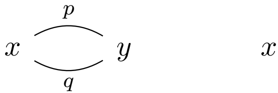
Paths are a kind of "equality" relating points (and paths) of a space.

E.g. The three presentations below are equivalent (the one on the right is a single point).



- α "says that" p and q are equal,
- p (or q) says that x and y are equal,
- so the space is just a single point.

However, the spaces below are *not* equivalent.



Although p, q both "say that" x and y are equal, nothing tells us that p and q are equal.

The Universe

There is a (big) space of all (small) spaces.

- Its points are spaces (or presentations of spaces),
- paths are equivalences between spaces,

. . . and so on.

Univalence

Paths in the universe tell us when two spaces are "the same".

So paths are a very good notion for "equality/equivalence" in space theory.

HoTT

The good news is that (a very good candidate for) space theory exists. It is **Homotopy Type Theory** (HoTT).

Everything is a type

The basic objects of space theory are called **types**. A type is a *formal space*.

For any elements x and y of a type S, the path type $\mathrm{Id}_S(x, y)$ is a type. So $\mathrm{Id}_S(-, -)$ is a **dependent type**.

Space theory *is* dependently typed, and so any structures defined in it are free to use dependent types in their definitions.

This thesis

Therefore, the broad goals of this thesis are to:

- set up a theory of dependently typed algebraic structures,
- set up a theory of their models in spaces,
- make a connexion with higher category theory via *locally presentable ∞-categories*.

Outline

Introduction and overview

Part 1: Dependently typed algebraic theories

- $\mathbf{C}$-contextual categories as monoids in collections
- Models of $\mathbf{C}$-contextual categories
- Homotopy models of $\mathbf{C}$-contextual categories

Part 2: Localisations of locally presentable ∞-categories

- Factorisation systems and pre-modulators
- (Lex) modalities and lex localisations

Conclusions and future work

A classical theorem of categorical algebra

Goal 1: Generalise the following result to dependent types.

Definition (Lawvere, Bénabou)

For $S \in \mathrm{Set}$, an S-**sorted algebraic theory** is a category with finite products whose objects are freely generated by S.

Theorem (Linton)

A multisorted algebraic theory is exactly the data of:

1. *A set S of sorts,*
2. *and a finitary monad on $\mathrm{Set}_{/S} \simeq \mathrm{Set}^S$.*

("finitary" = filtered-colimit preserving.)

Definition

A **model** of an S-sorted algebraic theory $\mathbf{T}$ is a finite-product-preserving functor $\mathbf{T} \to \mathrm{Set}$.

Theorem (Lawvere, Bénabou, Linton)

The category $\mathbf{T}\text{-}\mathrm{Mod}$ *of models of* $\mathbf{T}$ *is equivalent to the category of algebras of the associated finitary monad on* $\mathrm{Set}_{/S}$*.*

Thus $\mathbf{T}$ comes with a forgetful monadic functor $\mathbf{T}\text{-}\mathrm{Mod} \to \mathrm{Set}_{/S}$.

Examples

Monoids, groups, rings, modules, presheaves on a fixed category C, uncoloured operads etc.

Dependently typed/sorted algebraic theories

Many approaches

- Cartmell's generalised algebraic theories
- Makkai's logic with dependent sorts
- Fiore's Σ_n-models with substitution
- Palmgren's DFOL signatures
- Others (Aczel, Belo, QIITs ...)

but no monad/theory correspondence and recognition theorem.

We propose a *strictly less general* definition than each of these.

Type dependence $\sim$ Cellularity

Our definition is based on a fundamental duality between cellular structures and dependent types.

Objects of categories as "cells"

Every small category C has a relation on its objects:

$$c < d \qquad \text{iff} \qquad \text{there exists a non-identity morphism } c \to d$$

Definition

A small category C is **direct** if the relation $<$ is *well-founded*, i.e. there are no infinite descending chains $\ldots c_2 < c_1 < c_0$.

C is an **inverse** category if C^{op} is direct.

Type signatures

Definition

A **type signature** is a *locally finite* direct category.

A category C is **locally finite** if all of its slice categories are finite.

Equivalently, in every pullback in Cat of the form below

$$\begin{array}{ccc} A & \longrightarrow & C^{\rightarrow} \\ \big\downarrow & \lrcorner & \big\downarrow \mathrm{t} \\ 1 & \longrightarrow & C \end{array}$$

A is a finite category.

Remarks

- Objects of a direct category are formal "cells" and morphisms represent subcells.
- Objects of an inverse category are "dependent types" and morphisms represent type dependencies.
- Local finiteness says that every type depends on a *finite* context of variables.

Remark

$\mathbf{C}$ is locally finite direct iff $\mathbf{C}^{op}$ is **simple** *à la* Makkai's FOLDs.

Example

The direct category

$$\mathbb{G}_1 = \left\{ \; V \overset{s}{\underset{t}{\rightrightarrows}} E \; \right\}$$

can be written as the dependent type signature

$$\mathbf{S}_{\mathbb{G}_1} = \{ \vdash V \;\; \text{type} \quad , \quad x{:}V, y{:}V \vdash E(x,y) \;\; \text{type} \} \, .$$

A reasonable definition of a "model" of $\mathbf{S}_{\mathbb{G}_1}$ is

- a set X_V,
- and for every $(x, y) \in X_V \times X_V$, a set $X_E(x, y)$.

This is exactly a directed graph $X_E \rightrightarrows X_V$, which is a cellular structure.

Recognition theorem

Theorem (L.S., LeFanu Lumsdaine)

*A **dependently typed algebraic theory** is the data of*

1. *a locally finite direct category* $\mathbf{C}$ *(the type signature)*
2. *and a finitary monad (the algebraic theory) on* $\widehat{\mathbf{C}} \stackrel{\text{def}}{=} [\mathbf{C}^{op}, \mathrm{Set}]$*.*

Strictly generalising

Theorem (Linton)

A multisorted algebraic theory is exactly the data of:

1. *A set* S *of sorts,*
2. *and a finitary monad on* $\mathrm{Set}_{/S} \simeq \mathrm{Set}^S$*.*

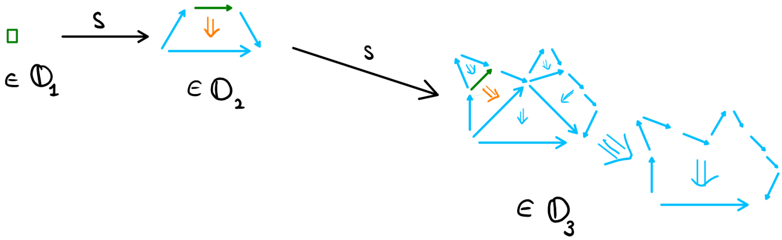
Examples of type signatures

1. Any set S (seen as a discrete category).
2. The ordinal ω (seen as a totally ordered poset).
3. The category $\mathbb{G}_1$.
4. The category $\mathbb{G}$ of *globes* :

$$D^0 \underset{t}{\overset{s}{\rightrightarrows}} D^1 \underset{t}{\overset{s}{\rightrightarrows}} D^2 \underset{t}{\overset{s}{\rightrightarrows}} \dots \quad ; \quad ss = ts \ , \ st = tt$$

Examples of type signatures

5. The category $\mathbb{O}$ of *opetopes* :



Examples of type signatures

6. Every Reedy category R has a wide direct subcategory R'. In most well-known examples, R' is also locally finite:
   - $R = \mathbf{\Delta}$, the *simplex* category ($\mathbf{\Delta}'$ = *semi-simplex* category),
   - $R = \mathbf{\Omega}_p$, the *planar dendroidal* category , ($\mathbf{\Omega}'_p$ = category of *planar semi-dendrices*),
   - $R = \mathbf{\Theta}$, Joyal's cell category .

   (in each case R' is the wide subcategory of monos.)
7. For $\mathbf{C}$ a locally finite direct category, for any $X \colon \mathbf{C}^{op} \to \mathrm{Set}$, its category of elements $\mathbf{C}/X$ is locally finite direct.

Examples of dependently typed algebraic theories

Using the recognition theorem, and the previous examples of type signatures, many well-known finitary monads correspond to dependently typed algebraic theories.

1. For $S \in \mathrm{Set}$, every S-sorted algebraic theory.
2. The identity monads on $\widehat{\mathbb{G}_1}$ (graphs), $\widehat{\mathbb{G}}$ (globular sets), $\widehat{\mathbb{O}}$ (opetopic sets), $\widehat{\mathbf{\Delta}'}$ (semi-simplicial sets).
3. The free-category monad on $\widehat{\mathbb{G}_1}$.
4. The free-strict-ω-category monad on $\widehat{\mathbb{G}}$.

Examples of dependently typed algebraic theories

5. For $T\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ a finitary cartesian monad, every T-operad (à la Burroni-Leinster) $T' \to T$.
6. Every free-*weak*-ω-category monad on $\widehat{\mathbb{G}}$.
7. and many more. . .

Let us now explain the recognition theorem.

A bit of category theory

Finite-colimit completions

- Let $\mathbf{C}$ be a small category.
- Write $\mathrm{Fin}_{\mathbf{C}}$ for the category of finitely presentable presheaves on $\mathbf{C}$. Write the dense inclusion as $E\colon \mathrm{Fin}_{\mathbf{C}} \hookrightarrow \widehat{\mathbf{C}}$.
- $\mathrm{Fin}_{\mathbf{C}}$ is the finite-colimit completion of $\mathbf{C}$. When $\mathbf{C}$ is a set, $\mathrm{Fin}_{\mathbf{C}}$ is the also the finite-coproduct completion of $\mathbf{C}$.

Cartesian collections

The presheaf category

$$\mathrm{Coll}_{\mathbf{C}} \stackrel{\mathsf{def}}{=} \left[\mathrm{Fin}_{\mathbf{C}}, \widehat{\mathbf{C}}\right] = [\mathrm{Fin}_{\mathbf{C}} \times \mathbf{C}^{op}, \mathrm{Set}]$$

is called the category of **cartesian $\mathbf{C}$-collections**. $(\mathrm{Coll}_{\mathbf{C}}, \circ, E)$ is a monoidal category (for a "substitution" product).

(Intuition: $F \in \mathrm{Coll}_{\mathbf{C}}$ should be thought of as a *term signature* — for each *context* $\Gamma \in \mathrm{Fin}_{\mathbf{C}}$ and each *sort* $s \in \mathbf{C}$, $F(s, \Gamma)$ is the set of "operations" with input Γ and output sort s.)

$\mathbf{C}$-sorted theories

Definition

A **$\mathbf{C}$-sorted theory** is an identity-on-objects, finite-colimit preserving functor $\mathrm{Fin}_{\mathbf{C}} \to \Theta$. A morphism is just a functor $\Theta \to \Theta'$ making the triangle commute. Call this category $\mathrm{Law}_{\mathbf{C}}$.

Fact (well-known)

$$\mathrm{Mon}(\mathrm{Coll}_{\mathbf{C}}, \circ, i) \;\simeq\; \mathrm{FinMnd}(\widehat{\mathbf{C}}) \;\simeq\; \mathrm{Law}_{\mathbf{C}}.$$

Remark

- We have only used that $\mathbf{C}$ is a small category.
- $\mathrm{FinMnd}(\widehat{\mathbf{C}})$ is the category of *monads with arities*, and $\mathrm{Law}_{\mathbf{C}}$ is the category of *Lawvere theories with arities*, for the arities $E\colon \mathrm{Fin}_{\mathbf{C}} \hookrightarrow \widehat{\mathbf{C}}$.

Back to dependent types

Contexts as cell complexes

Let $\mathbf{C}$ once again be a type signature (a locally finite direct category).

Theorem (L.S., LeFanu Lumsdaine)

$\mathrm{Fin}^{op}_{\mathbf{C}}$ is equivalent to a contextual category $\mathrm{Cx}(\mathbf{C})$, which is the free contextual category on $\mathbf{C}$.

- Particular case: $\mathbf{C}$ is a set, then $\mathrm{Fin}^{op}_{\mathbf{C}}$ is the free finite-product category on $\mathbf{C}$.

Remark

The structure of a contextual category *does not* transfer across an equivalence of categories.

Define a **cell context** to be a finite sequence in $\widehat{\mathbf{C}}$

$$\emptyset \to X_1 \to X_2 \to \ldots \to X$$

of *chosen* pushouts of boundary inclusions of representables:

$$\begin{array}{ccc} \partial c & \longrightarrow & X_n \\ \big\downarrow & \ulcorner & \big\downarrow \\ c & \longrightarrow & X_{n+1}. \end{array}$$

Definition

The category $\mathrm{Cell}_{\mathbf{C}}$ has as objects the cell contexts and as morphisms, $\mathrm{Cell}_{\mathbf{C}}(\emptyset \ldots \to X, \emptyset \ldots \to Y) \stackrel{\text{def}}{=} \widehat{\mathbf{C}}(X, Y)$.

Then $\mathrm{Cell}_{\mathbf{C}} \simeq \mathrm{Fin}_{\mathbf{C}}$ and we have that $\mathrm{Cx}(\mathbf{C}) \stackrel{\text{def}}{=} \mathrm{Cell}_{\mathbf{C}}^{op}$ is the free contextual category on $\mathbf{C}$.

Dependently typed algebraic theories

A **$\mathbf{C}$-contextual category** is a morphism $f\colon \mathsf{Cx}(\mathbf{C}) \to \mathsf{D}$ in CxlCat whose (id.-on-objects, f.f.) factorisation

$$\mathsf{Cx}(\mathbf{C}) \xrightarrow{j_f} \Theta_{\mathsf{D}} \hookrightarrow \mathsf{D}$$

is such that for every map $g\colon \mathsf{Cx}(\mathbf{C}) \to \mathsf{D}'$ in CxlCat and diagram

$$\begin{array}{ccccc} \mathsf{Cx}(\mathbf{C}) & \xrightarrow{j_f^{\mathsf{op}}} & \Theta_f^{\mathsf{op}} & \hookrightarrow & \mathsf{D} \\ & {\scriptstyle g} \searrow & \downarrow{\scriptstyle h} & \swarrow {\scriptstyle \exists! \tilde{h}} & \\ & & \mathsf{D}' & & \end{array}$$

(where h is any functor), $\exists! \tilde{h}$ in CxlCat making the diagram commute. A morphism of $\mathbf{C}$-contextual categories is just a morphism in the coslice $\mathsf{Cx}(\mathbf{C})/\mathrm{CxlCat}$.

Classification of dependently sorted algebraic theories

Theorem (L.S., LeFanu Lumsdaine)

Given a type signature $\mathbf{C}$*, the categories*

1. $\mathrm{Mon}(\mathrm{Coll}_{\mathbf{C}})$ *of monoids in cartesian* $\mathbf{C}$*-collections,*
2. $\mathrm{FinMnd}(\widehat{\mathbf{C}})$ *of finitary monads on* $\widehat{\mathbf{C}}$*,*
3. $\mathrm{Law}_{\mathbf{C}}$ *of* $\mathbf{C}$*-sorted theories,*
4. *and* $\mathrm{CxlCat}_{\mathbf{C}}$ *of* $\mathbf{C}$*-contextual categories,*

are equivalent.

Set-valued models

Definition

A **(Set-)model** of a $\mathbf{C}$-contextual category $\mathsf{Cx}(\mathbf{C}) \to \mathsf{D}$ is a presheaf $X \colon \mathsf{D} \to \mathrm{Set}$ such that the composite $\mathsf{Cx}(\mathbf{C}) \to \mathsf{D} \xrightarrow{X} \mathrm{Set}$

1. takes $\emptyset \in \mathrm{Cell}_{\mathbf{C}}$ to $1 \in \mathrm{Set}$,
2. and takes every chosen pushout

$$\begin{array}{ccc} \partial c & \longrightarrow & X_n \\ \big\downarrow & \ulcorner & \big\downarrow \\ c & \longrightarrow & X_{n+1} \end{array}$$

in $\mathrm{Cell}_{\mathbf{C}}$ to a pullback square in Set.

A morphism of models is just a natural transformation.

Models and algebras

Theorem (L.S.)

The category of models of a $\mathbf{C}$*-contextual category* $\mathrm{Cx}(\mathbf{C}) \to \mathsf{D}$ *is equivalent to:*

1. *the category of algebras of the associated finitary monad on* $\widehat{\mathbf{C}}$*,*
2. *the category of* Set*-models of the contextual category* D*.*

Theorem (L.S.)

$\mathcal{C}$ *is a locally finitely presentable category iff it is the category of models of a* $\mathbf{C}$*-contextual category for some type signature* $\mathbf{C}$*.*

(Thus $\mathbf{C}$-contextual categories are *Morita equivalent* to essentially algebraic theories/finite-limit theories.)

Proof

1. Every category of models of a $\mathbf{C}$-contextual category is a category of models of a finite-limit sketch, so is locally finitely presentable.

2. Consider the non-full inclusion $i_{\Delta'} \colon \mathbf{\Delta}' \to \mathrm{Cat}$.

   For $A \in \mathrm{Cat}$, let $\mathbf{\Delta}'/A$ be the comma-category. Then:

   - $i_{\Delta'}$ has an associated *nerve functor* $\nu \colon \mathrm{Cat} \to \widehat{\mathbf{\Delta}'}$,
   - and $\mathbf{\Delta}'/A$ is the category of elements of $\nu(A)$.

   Thus $\mathbf{\Delta}'/A$ is a locally finite direct category.

   Consider $\tau_A \colon \mathbf{\Delta}'/A \to A$ taking $\{0 < \ldots < n\} \xrightarrow{f} A$ to $f(n)$.

   (Cisinski) The pullback functor $\tau_A^* \colon \widehat{A} \hookrightarrow \widehat{\mathbf{\Delta}'/A}$ is fully faithful.

3. Every locally finitely presentable category $\mathcal{C}$ has a filtered-colimit preserving, fully faithful right adjoint $\mathcal{C} \hookrightarrow \widehat{A}$ to a presheaf category. Then the composite

$$\mathcal{C} \hookrightarrow \widehat{A} \xhookrightarrow{\tau_A^*} \widehat{\mathbf{\Delta}'/A}$$

is fully faithful, monadic and filtered-colimit preserving. $\square$

Homotopical $\mathbf{C}$-spaces

Definition

A **homotopical $\mathbf{C}$-space** is a simplicial presheaf $F\colon \mathrm{Cell}_{\mathbf{C}}^{op} \to \mathrm{sSet}$

1. such that $F\emptyset$ is contractible,
2. and F takes every chosen pushout

$$\begin{array}{ccc} \partial c & \longrightarrow & X_n \\ \big\downarrow & \ulcorner & \big\downarrow \\ c & \longrightarrow & X_{n+1} \end{array}$$

to a homotopy pullback square, i.e. $FX_{n+1} \simeq FX_n \times^h_{F\partial c} Fc$.

Homotopical models

Definition

A **homotopical model** of a $\mathbf{C}$-contextual category $\mathrm{Cx}(\mathbf{C}) \to \mathsf{D}$ is a simplicial presheaf $\mathsf{D} \to \mathrm{sSet}$ such that $\mathrm{Cx}(\mathbf{C}) \to \mathsf{D} \to \mathrm{sSet}$ is a homotopical $\mathbf{C}$-space.

Remark

Pullbacks in sSet are *not* homotopy limits, so we cannot reformulate this condition by requiring that the canonical map $FX_{n+1} \to FX_n \times_{F\partial c} Fc$ to the strict pullback be a weak equivalence.

Flasque model structure

Due to this subtlety, we introduce a global model structure on the simplicial presheaf category $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}}) \overset{\mathsf{def}}{=} \left[\mathrm{Cell}_{\mathbf{C}}^{op}, \mathrm{sSet}\right]$.

Flasque boundaries

For $c \in \mathbf{C}$, let "∂c" be the colimit of the composite

$$\mathbf{C}^{-}_{/c} \to \mathbf{C} \hookrightarrow \mathrm{Cell}_{\mathbf{C}} \hookrightarrow \widehat{\mathrm{Cell}_{\mathbf{C}}}.$$

We have a composite inclusion in $\widehat{\mathrm{Cell}_{\mathbf{C}}}$

$$\text{``}\delta_c\text{''} \colon \text{``}\partial c\text{''} \longhookrightarrow \partial c \xhookrightarrow{\delta_c} c$$

where $\partial c \hookrightarrow c$ is representable in $\mathrm{Cell}_{\mathbf{C}}$.

Definition

A map $p\colon X \to Y$ in $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})$ is a ∂-**flasque fibration** if the "pullback-hom" map

$$\langle\text{``}\delta_c\text{''}, p\rangle : X_c \longrightarrow \mathrm{Map}(\text{``}\partial c\text{''}, X) \times_{\mathrm{Map}(\text{``}\partial c\text{''}, Y)} Y_c$$

in sSet is a Kan fibration.

Theorem (L.S.)

*The **flasque** model structure on* $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})$ *whose weak equivalences are the global (objectwise) weak equivalences, and whose fibrations are the ∂-flasque fibrations, exists. We write it* $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})_{\partial}$.

Remarks

1. $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})_\partial$ is *intermediate*: the identity functor gives Quillen equivalences

$$\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})_{\textit{proj}} \rightleftarrows \mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})_\partial \rightleftarrows \mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})_{\textit{inj}}$$

   where (*proj* = projective) and (*inj* = injective) model structures.

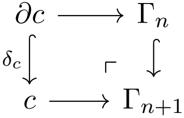
2. For the inclusion $i\colon \mathbf{C} \hookrightarrow \mathrm{Cell}_{\mathbf{C}}$, both adjunctions

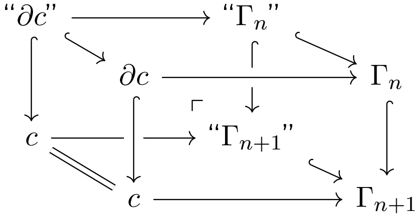
$$\mathrm{Sp}(\mathbf{C})_{\textit{inj}} \underset{i_*}{\overset{i_!}{\substack{\hookrightarrow \\ \xleftarrow{\;i^*\;} \\ \hookrightarrow}}} \mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})_\partial$$

   are Quillen for the injective Reedy model structure $\mathrm{Sp}(\mathbf{C})_{\textit{inj}}$.

Model structure for homotopy $\mathbf{C}$-spaces

For every object of $\mathrm{Cell}_{\mathbf{C}}$ (a finite cell complex $\emptyset \to \Gamma_1 \to \ldots \to \Gamma$) we inductively define the subrepresentable "Γ" $\hookrightarrow \Gamma$ in $\widehat{\mathrm{Cell}_{\mathbf{C}}}$, by defining "$\emptyset$" to be the empty presheaf and by:





Definition

The **model structure for homotopy $\mathbf{C}$-spaces** is the left Bousfield localisation of $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})_{\partial}$ at the set of maps

$$S_{\partial} \stackrel{\text{def}}{=} \{s_{\Gamma} \colon \text{``}\Gamma\text{''} \hookrightarrow \Gamma \mid \Gamma \in \mathrm{Cell}_{\mathbf{C}}\}.$$

We write it as $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})_{\partial}^{l}$.

Fibrant objects of $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})_{\partial}^{l}$ are called **homotopy $\mathbf{C}$-spaces**.

Recall

X is a fibrant object of $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})_{\partial}^{l}$ iff it is S_{∂}-**local** : i.e. it is fibrant in $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})_{\partial}$ and every $\langle s_{\Gamma}, X\rangle : X_{\Gamma} \to \mathrm{Map}(\text{``}\Gamma\text{''}, X)$ is a weak equivalence in sSet.

Theorem (L.S.)

The adjunction $i^* : \mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})^l_\partial \rightleftarrows \mathrm{Sp}(\mathbf{C})_{inj} : i_*$ *is a Quillen equivalence.*

Thus $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})^l_\partial$ presents the presheaf ∞-category $\mathcal{P}(\mathbf{C})$.

Theorem (L.S.)

If X *is fibrant in* $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})^l_\partial$*, then it is a homotopical* $\mathbf{C}$*-space.*

Homotopy models of any $\mathbf{C}$-contextual category

$\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})^{l}_{\partial}$ is the model structure for homotopy models of the **initial** $\mathbf{C}$-contextual category.

For an arbitrary $\mathbf{C}$-contextual category $\mathrm{Cx}(\mathbf{C}) \to \mathsf{D}$, we consider the (id-on-objects, f.f.) factorisation $\mathrm{Cell}_{\mathbf{C}} \xrightarrow{j} \Theta_{\mathsf{D}} \hookrightarrow \mathsf{D}^{op}$.

Theorem (L.S.)

The right-transferred model structure along $j^{*} \colon \mathrm{Sp}(\Theta_{\mathsf{D}}) \to \mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})_{\partial}$ *exists.*

We call it (too) the ∂-**flasque** model structure and write $\mathrm{Sp}(\Theta_{\mathsf{D}})_{\partial}$.

Definition

The **model category of homotopy** D-**algebras** is the left Bousfield localisation of the model category $\mathrm{Sp}(\Theta_{\mathsf{D}})_\partial$ at the set of maps $j_! S_\partial$. We write it as $\mathrm{Sp}(\Theta_{\mathsf{D}})^l_\partial$.

Corollary

The adjunction $j_! : \mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})^l_\partial \rightleftarrows \mathrm{Sp}(\Theta_{\mathsf{D}})^l_\partial : j^*$ *is Quillen.*

Remark

$\mathrm{Sp}(\Theta_{\mathsf{D}})^l_\partial$ presents a locally presentable ∞-category.

Rigidification

Conjecture (Rigidification for $\mathbf{C}$-contextual categories)

The adjunction $\mathrm{Sp}(\Theta_{\mathsf{D}})^{l}_{\partial} \rightleftarrows \mathrm{s}\mathsf{D}\text{-}\mathrm{Mod}$ is a (Quillen or DK) equivalence.

- This is true in the particular casewhen $\mathbf{C}$ is a set (Badzioch).
- We have seen that this is true for the initial $\mathbf{C}$-contextual category.

Implicit ∞-convention

We work with the category $\mathcal{S}$ of spaces, with "∞" everywhere.

We write $[x, y]_{\mathcal{C}}$ or just $[x, y]$ for hom-spaces of any $\mathcal{C}$.

Orthogonality

Let $l, r \in \mathcal{C}^{\to}$. We write $l \perp r$ if every square in $\mathcal{C}$

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle l}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle r} \\ B & \longrightarrow & Y \end{array}$$

has a unique lift. Equivalently, the pullback-hom map in $\mathcal{S}$

$$\langle l, r \rangle : [B, X]_{\mathcal{C}} \to [l, r]_{\mathcal{C}^{\to}}$$

is invertible.

Orthogonal factorisation systems

Given a full subcategory $W \hookrightarrow \mathcal{C}^{\rightarrow}$, we define ${}^{\perp}W, W^{\perp} \hookrightarrow \mathcal{C}^{\rightarrow}$ in the usual way. An **orthogonal system** is a pair $(\mathcal{L}, \mathcal{R})$ of full subcategories of $\mathcal{C}^{\rightarrow}$ such that $\mathcal{L} = {}^{\perp}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\perp}$.

For every $W \hookrightarrow \mathcal{C}^{\rightarrow}$, $({}^{\perp}(W^{\perp}), W^{\perp})$ is an orthogonal system.

Factorisation systems

An orthogonal system $(\mathcal{L}, \mathcal{R})$ is a **factorisation system (FS)** if every f in $\mathcal{C}^{\rightarrow}$ factors as $f = pi$ with $i \in \mathcal{L}$ and $p \in \mathcal{R}$.

Accessible FS

Theorem (Gabriel–Ulmer, Kelly)

Let $\mathcal{C}$ be a locally presentable 1-category and let $W \hookrightarrow \mathcal{C}^{\to}$ be a small subcategory. Then $({}^{\perp}(W^{\perp}), W^{\perp})$ is a FS.

Theorem (Anel, L.S.)

Let $\mathcal{C}$ be a locally presentable category and let $W \hookrightarrow \mathcal{C}^{\to}$ be a small subcategory. Then $({}^{\perp}(W^{\perp}), W^{\perp})$ is a FS.

Remark

Requires a small modification of Kelly's construction, but a more conceptual understanding in terms of the tensor/enrichment of $\mathcal{C}^{\to}$ over $\mathcal{S}^{\to}$ given by the pushout-product/pullback-hom.

Sheafification

Suppose C is a Grothendieck site, and let $W \hookrightarrow \mathcal{P}(C)^{\rightarrow}$ be the full subcategory of covering sieves.

Then the factorisation of $X \to 1$ in $({}^{\perp}(W^{\perp}), W^{\perp})$ is the sheafification of X.

Plus-construction

Consider the functor $X \mapsto X^{+}$ where

$$X^{+} \stackrel{\text{def}}{=} \int^{w \in W} [\mathrm{s}w, X] \times \mathrm{t}w.$$

Then it is well known that a transfinite iteration $X^{+\omega}$ converges to the sheafification of X (and in the 1-topos $\widehat{C}$, two iterations suffice).

Pre-modulators

The plus-construction is much simpler than Kelly's construction; we would like to use it for all FS.

Definition

Let $\mathcal{C}$ be locally presentable, with a small full subcategory $C \subset \mathcal{C}$ of generators. A **pre-modulator** is a full subcategory $W \hookrightarrow \mathcal{C}^{\to}$ such that

- the codomains of the maps in W are all in C,
- the inclusion $C \hookrightarrow \mathcal{C}^{\to}$ sending a generator to its identity map factors through $W \hookrightarrow \mathcal{C}^{\to}$.

General plus-construction

Theorem (Anel, L.S.)

Let $W \hookrightarrow \mathcal{C}^{\to}$ be a pre-modulator and let f in $\mathcal{C}^{\to}$. Then a transfinite iteration of the plus-construction

$$f \mapsto f^{+} \stackrel{\text{def}}{=} \operatorname*{colim}_{W \downarrow f} \mathrm{t}w \to \operatorname*{colim}_{W \downarrow_{\mathrm{t}} \mathrm{t}f} \mathrm{t}w$$

converges to the factorisation of f in $({}^{\perp}(W^{\perp}), W^{\perp})$.

Theorem (Anel, L.S.)

Every small diagram $W \hookrightarrow \mathcal{C}^{\to}$ can be completed into a pre-modulator generating the same orthogonal system. Thus every accessible FS on a locally presentable category is generated by iterating the plus-construction.

(Lex) modalities and modulators

A FS $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ is a **modality** if $\mathcal{L}$ is stable under base change along any map. $(\mathcal{L}, \mathcal{R})$ is a **lex modality** if $\mathcal{L}$ is stable under finite limits in $\mathcal{C}^{\to}$.

Definition

A pre-modulator $W \hookrightarrow \mathcal{C}^{\to}$ is a **modulator** if the codomain functor $\mathrm{t}\colon W \to C$ is a fibration.

A modulator is **lex** if the fibres of $\mathrm{t}\colon W \to C$ are cofiltered.

Theorem (Anel, L.S.)

Let $\mathcal{C}$ be locally presentable and locally cartesian closed (resp. a topos). The plus-construction associated to a (lex) modulator generates a (lex) modality, and every accessible (lex) modality on $\mathcal{C}$ is generated by a (lex) modulator.

Theorem (Anel, L.S.)

Let $W \hookrightarrow \mathcal{E}^{\rightarrow}$ be a ***mono-saturated*** *lex modulator on a topos. Then the plus-construction converges in $(n+2)$ steps on every n-truncated object/map of $\mathcal{E}$.*

Introduction and overview

Part 1: Dependently typed algebraic theories

$\mathbb{C}$-contextual categories as monoids in collections

Models of $\mathbb{C}$-contextual categories

Homotopy models of $\mathbb{C}$-contextual categories

Part 2: Localisations of locally presentable ∞-categories

Factorisation systems and pre-modulators

(Lex) modalities and lex localisations

Conclusions and future work

To sum up

Part 1:

- Define and classify dependently typed algebraic theories
- Define their models and homotopy-models, state a rigidification conjecture and prove it in a degenerate case.
- Introduce opetopic theories and opetopic algebras, and prove the rigidification conjecture for opetopic theories.

Part 2: (∞-category theory)

- Define pre-modulators and prove a correspondence with accessible factorisation systems
- Prove a correspondence between (lex) modulators and (lex) modalities.

What I'm thinking about

1. Rigidification conjecture using *weak* model structures (thanks to S. Henry).
2. Dependently coloured operads :
   Coloured operads $\rightsquigarrow$ algebraic theories
   vs. ??? $\rightsquigarrow$ dependently typed algebraic theories.
3. *Polygraphs* (contexts) of a $\mathbf{C}$-contextual category D, and their relation to generic-free factorisations in D.
4. Higher algebraic theories and locally presentable ∞-categories.

Thank you!

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Annexes

Enter logic

Equality in set theory is a **predicate**.

In traditional logic, predicates are kept distinct from the sets they talk about (which are called **sorts**).

Example

- "is greater than" is a predicate (not a set)
- 3 and 4 are elements of a set (natural numbers) to which we can *apply* "is greater than"
- to get the sentence "3 is greater than 4".

Logic for space theory

Problem

Path spaces encode equality (a predicate), but they are also spaces.

Solution

Define “space theory” in a logical framework where

there is no difference between predicates and sets/sorts.

Such as **Martin-Löf type theory**.

A 21st-century breakthrough

Homotopy Type Theory (HoTT), based on Martin-Löf type theory has been shown to be a very good candidate for the formal theory of spaces.

HoTT is defined in the logical framework of dependent types.

So (in a sense) the problem of defining "space theory" has been solved.

Algebra in spaces

Algebraic structures in spaces are called **higher algebraic structures**.

We use set theory to define the operation $+$ on the set of natural numbers, and to prove that $(\mathbb{N}, +)$ is an associative monoid ...

... we would like to use HoTT to define the operation of composition on a loop space $\mathrm{Id}_S(x, x)$, and to prove that it is an associative monoid up to *coherent paths*.

Dependently typed algebraic structures

Problem

To tackle higher algebraic structures in HoTT, we need to understand the ordinary ("lower") algebraic structures defined by dependently typed theories.

Whence this thesis.

Overview of results

Part 1

Dependently typed algebraic theories

- Define **dependently typed/sorted algebraic theories** and prove a classification ("recognition") theorem.
- Define Set-models of dependently typed algebraic theories, and prove a Morita equivalence with essentially algebraic/finite-limit theories.
- Define space-valued models of dependently typed algebraic theories via a local model structure on simplicial presheaves.
- Conjecture a general rigidification theorem w.r.t. strict simplicial algebras, and prove a very degenerate case.

Overview of results

Opetopic theories

- Define a family of dependently typed algebraic theories typed by the category $\mathbb{O}$ of opetopes, and prove a general nerve theorem for each.
- Show a rigidification theorem for space-valued models of each opetopic theory.

Overview of results

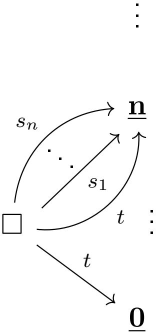
Part 2

Localisations of presentable ∞-categories

- Define **pre-modulators** and prove they generate all accessible factorisation systems in presentable ∞-categories via a **plus-construction**.
- Use the theory of pre-modulators to define (**lex**) **modulators** and show they correspond exactly to (**lex**) **modalities**.

Examples of type signatures

3. The category $\mathrm{el}\mathbb{T}\mathrm{r}_p$ of *planar elementary trees* or *planar corollas*:



Recall: Let α be a *cartesian* natural transformation. Then G preserves every colimit that F does. So F finitary $\Rightarrow G$ finitary.

$$\widehat{\mathbf{C}} \ \underset{F}{\overset{G}{\Downarrow \alpha}} \ \widehat{\mathbf{C}}$$

Examples of dependently typed algebraic theories

4. The free-planar-coloured-Set-operad monad on $\widehat{\mathrm{el}\mathbb{T}\mathrm{r}_p}$ (planar coloured collections).
5. The free-simplicial-set monad on $\widehat{\mathbf{\Delta}'}$.
6. And many more (Morita equivalence with essentially algebraic/finite-limit theories)...

Composition of cartesian collections

$\mathbf{C}$-collections can be composed via **substitution**:

$$G \circ F(i, \Gamma) \overset{\mathsf{def}}{=} \int^{\Theta \in \mathrm{Fin}_{\mathbf{C}}} G(i, \Theta) \times \widehat{\mathbf{C}}(\Theta, F(-, \Gamma)).$$

$(\mathrm{Coll}_{\mathbf{C}}, \circ, E)$ is a **monoidal category**, where $E : \mathrm{Fin}_{\mathbf{C}} \hookrightarrow \widehat{\mathbf{C}}$.

Cartesian collections and endofunctors on $\widehat{\mathbf{C}}$

The functor $\mathrm{Lan}_E(-) : \mathrm{Coll}_{\mathbf{C}} \hookrightarrow \left[\widehat{\mathbf{C}}, \widehat{\mathbf{C}}\right]$ of left Kan extension along $E : \mathrm{Fin}_{\mathbf{C}} \hookrightarrow \widehat{\mathbf{C}}$ is (1) **fully faithful** and (2) **monoidal**.

$$\begin{array}{ccc} \mathrm{Fin}_{\mathbf{C}} & \xrightarrow{F} & \widehat{\mathbf{C}} \\ {\scriptstyle E}\big\downarrow & \cong & \nearrow{\scriptstyle \mathrm{Lan}_E F} \\ \widehat{\mathbf{C}} & & \end{array} \qquad (1)$$

$$\mathrm{Lan}_E(F \circ G) \;\cong\; \mathrm{Lan}_E F \circ \mathrm{Lan}_E G \quad ; \quad \mathrm{Lan}_E E \;\cong\; \mathrm{id} \qquad (2)$$

Consequence

$$\mathrm{Lan}_E - : \mathrm{Mon}(\mathrm{Coll}_{\mathbf{C}}, \circ, E) \hookrightarrow \mathrm{Mnd}(\widehat{\mathbf{C}})$$

The category of monoids in $\mathrm{Coll}_{\mathbf{C}}$ is a full subcategory of the category of monads on $\widehat{\mathbf{C}}$. It is none other than the category of **finitary monads** on $\widehat{\mathbf{C}}$.

Proof sketch

1. First, we define :

- For c in $\mathbf{C}$, let $\mathbf{C}_{/c}^{-} \subset \mathbf{C}_{/c}$ be the full subcategory obtained by removing the terminal object of $\mathbf{C}_{/c}$ (the identity morphism).
- The **boundary** ∂c of c is the presheaf that is the colimit of

$$\mathbf{C}_{/c}^{-} \to \mathbf{C}_{/c} \to \mathbf{C} \hookrightarrow \widehat{\mathbf{C}}.$$

- We have an inclusion $\partial c \hookrightarrow c$ in $\widehat{\mathbf{C}}$ (where c is the representable presheaf).

  A **finite cell complex** is a finite sequence $\emptyset \to X_1 \to \ldots \to X$ of pushouts of boundary inclusions.

2. Next, note that:

- The Yoneda embedding factors as $\mathbf{C} \hookrightarrow \mathrm{Fin}_{\mathbf{C}} \hookrightarrow \widehat{\mathbf{C}}$.
- The boundary inclusions $\partial c \hookrightarrow c$ are finitely presentable (since $\mathbf{C}_{/c}$ is finite).
- Every finite cell complex in $\widehat{\mathbf{C}}$ is finitely presentable.
- Every $X \in \mathrm{Fin}_{\mathbf{C}}$ can be written as a finite cell complex.

3. Define a **cell context** to be a finite sequence

$$\emptyset \to X_1 \to X_2 \to \ldots \to X$$

of *chosen* pushouts of boundary inclusions:

$$\begin{array}{ccc} \partial c & \longrightarrow & X_n \\ \big\downarrow & \ulcorner & \big\downarrow \\ c & \longrightarrow & X_{n+1}. \end{array}$$

Definition

The category $\mathrm{Cell}_{\mathbf{C}}$ has as objects the cell contexts and as morphisms, $\mathrm{Cell}_{\mathbf{C}}(\emptyset \ldots \to X, \emptyset \ldots \to Y) \stackrel{\text{def}}{=} \widehat{\mathbf{C}}(X, Y)$.

Clearly, $\mathrm{Cell}_{\mathbf{C}} \simeq \mathrm{Fin}_{\mathbf{C}}$.

4. Not hard to see that $\mathrm{Cx}(\mathbf{C}) \stackrel{\text{def}}{=} \mathrm{Cell}_{\mathbf{C}}^{op}$ is a contextual category. (In fact, it is the *free contextual category on* $\mathbf{C}$.) □

Remarks

- A collection

$$X \in \mathrm{Coll}_{\mathbf{C}} \simeq \left[\mathrm{Cell}_{\mathbf{C}}, \widehat{\mathbf{C}}\right] = \mathrm{Set}^{(\mathrm{Cx}(\mathbf{C}) \times \mathbf{C})^{op}}$$

  is now *literally* a $\mathbf{C}$-sorted term signature.

- $\mathrm{Cx}(\mathbf{C})$ has all finite limits.

Morita equivalence with essentially algebraic theories

Theorem (L.S.)

Every locally finitely presentable category $\mathcal{C}$ is the category of models of a $\mathbf{C}$-contextual category for some type signature $\mathbf{C}$.

Proof

1. Every category of models of a $\mathbf{C}$-contextual category is a category of models of a finite-limit sketch, so is locally finitely presentable.

2. Consider the non-full inclusion $i_{\Delta'}\colon \mathbf{\Delta}' \to \mathrm{Cat}$.

For $A \in \mathrm{Cat}$, let $\mathbf{\Delta}'/A$ be the comma-category. Then:

- $i_{\Delta'}$ has an associated *nerve functor* $\nu\colon \mathrm{Cat} \to \widehat{\mathbf{\Delta}'}$,
- and $\mathbf{\Delta}'/A$ is the category of elements of $\nu(A)$.

Thus $\mathbf{\Delta}'/A$ is a locally finite direct category.

Consider $\tau_A\colon \mathbf{\Delta}'/A \to A$ taking $\{0 < \ldots < n\} \xrightarrow{f} A$ to $f(n)$.

(Cisinski) The pullback functor $\tau_A^*\colon \widehat{A} \hookrightarrow \widehat{\mathbf{\Delta}'/A}$ is fully faithful.

3. Every locally finitely presentable category $\mathcal{C}$ has a filtered-colimit preserving, fully faithful right adjoint $\mathcal{C} \hookrightarrow \widehat{A}$ to a presheaf category. Then the composite

$$\mathcal{C} \hookrightarrow \widehat{A} \xhookrightarrow{\tau_A^*} \widehat{\mathbf{\Delta}'/A}$$

is fully faithful, monadic and filtered-colimit preserving. □

Homotopy models

Definition

A **homotopy model** of a $\mathbf{C}$-contextual category $\mathrm{Cx}(\mathbf{C}) \to \mathsf{D}$ is a presheaf $\mathsf{D} \to \mathcal{S}$ to the ∞-category of spaces, such that the composite $\mathrm{Cx}(\mathbf{C}) \to \mathsf{D} \to \mathcal{S}$

1. takes $\emptyset \in \mathrm{Cell}_{\mathbf{C}}$ to the terminal space $1 \in \mathcal{S}$,
2. and takes every chosen pushout

$$\begin{array}{ccc} \partial c & \longrightarrow & X_n \\ \big\downarrow & \ulcorner & \big\downarrow \\ c & \longrightarrow & X_{n+1} \end{array}$$

in $\mathrm{Cell}_{\mathbf{C}}$ to a pullback square in the ∞-category $\mathcal{S}$.

Models of algebraic theories in spaces

When $\mathbf{C}$ is a set, then $\mathrm{Cx}(\mathbf{C}) \to \mathsf{D}$ is an identity-on-objects, finite-product preserving functor.

Thus $\mathsf{D} \to \mathcal{S}$ is a homotopy-model iff it preserves finite products.

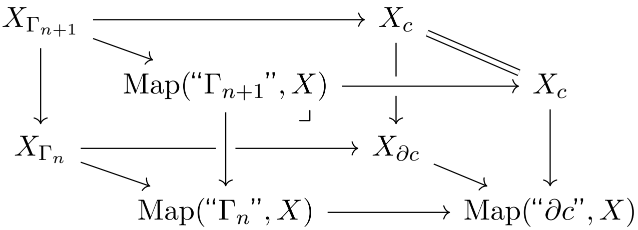
This can be represented by a **simplicial presheaf** $X \colon \mathsf{D} \to \mathrm{sSet}$ that takes finite products in D to *homotopy limits*.

Products in sSet are already homotopy limits, so this condition can be formulated as:

For every $\Gamma = c_1 \times \ldots \times c_k$ in D, the canonical map $X\Gamma \to Xc_1 \times \ldots \times Xc_k$ is a weak equivalence in sSet.

Proof

For any X in $\mathrm{Sp}(\mathrm{Cell}_{\mathbf{C}})$, we have the cube in sSet



whose front face is cartesian. If X is S_∂-local, then

- all corners of the cube are fibrant objects,
- $X_c \to \mathrm{Map}(``\partial c", X)$ is a fibration,
- and the intervening arrows are weak equivalences.

So the front and back faces are homotopy pullbacks. □