Dependently typed theories as “cellular” Lawvere theories

Work in progress

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Dependent type theory vs. dependently typed theories

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Dependent type theory (DTT) is a **language**:
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A **dependently typed theory** is a **theory**:
it is a set of types, terms and equalities expressed in DTT.

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Rules of DTT

$$\frac{}{\vdash \diamond \ \mathrm{ctxt}}\ \texttt{EMP} \qquad \frac{\Gamma \vdash A \ \mathrm{type}}{\vdash \Gamma, x : A \ \mathrm{ctxt}}\ \texttt{EXT}$$

$$\frac{\vdash \Gamma, x : A, \Delta \ \mathrm{ctxt}}{\Gamma, x : A, \Delta \vdash x : A}\ \texttt{VAR}$$

$$\frac{\Gamma, \Delta \vdash \mathcal{J} \qquad \Gamma \vdash A \ \ \mathrm{type}}{\Gamma, x : A, \Delta \vdash \mathcal{J}}\ \texttt{WEAK}$$

$$\frac{\Gamma, x : A, \Delta \vdash \mathcal{J} \quad \Gamma \vdash a : A}{\Gamma, \Delta[a / x] \vdash \mathcal{J}[a / x]}\ \texttt{SUBST}$$

. . . not important for the rest of the talk.

"Baby" example of a dependently typed theory

"Baby" theories don't have type dependency — e.g. the theory of **abelian groups**:

$$\vdash G \text{ type}$$

$$\vdash 1 : G$$

$$x : G \vdash x^{-1} : G$$

$$x, y : G \vdash x \cdot y : G$$

$$x, y, z : G \vdash (x \cdot y) \cdot z = x \cdot (y \cdot z) : G$$

$$x, y : G \vdash x \cdot y = y \cdot x : G$$

$$x : G \vdash x \cdot 1 = x : G$$

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also of monoids, groups, rings, modules, algebras, Lie algebras, bialgebras, Hopf algebras ...

"Adult" example of a dependently typed theory

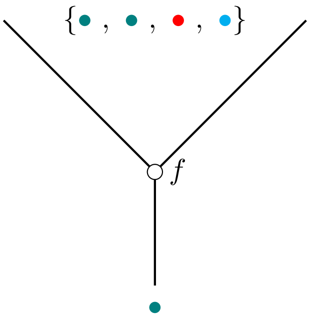
The theory of **categories**:

$$\begin{aligned} &\vdash \mathrm{Ob}\ \mathrm{type} \\ x, y : \mathrm{Ob} &\vdash \mathrm{Hom}(x, y)\ \mathrm{type} \\ x : \mathrm{Ob} &\vdash 1_x : \mathrm{Hom}(x, x) \\ \ldots, f : \mathrm{Hom}(x, y), g : \mathrm{Hom}(y, z) &\vdash g \circ f : \mathrm{Hom}(x, z) \\ x, y : \mathrm{Ob}, f : \mathrm{Hom}(x, y) &\vdash f \circ 1_x = f = 1_y \circ f : \mathrm{Hom}(x, y) \\ \ldots &\vdash (h \circ g) \circ f = h \circ (g \circ f) : \mathrm{Hom}(x_1, x_4) \end{aligned}$$

also of 2-categories, ω-categories, reflexive graphs, semisimplicial sets, opetopic sets . . .

Sneak peek at the big picture

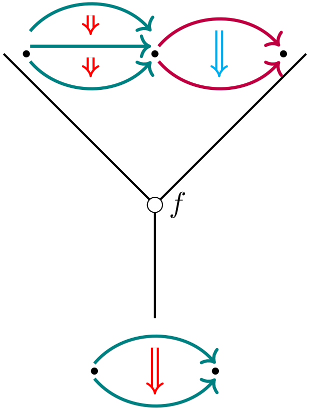
Operations in a "baby" theory



An operation in a multisorted Lawvere theory takes a finite coproduct of points as input, and outputs a point.

Sneak peek at the big picture

Operations in an "adult" theory



Every operation in a dependently typed theory takes a finite cell complex as input, and outputs a cell.

(This is related to Burroni-Leinster T-operads.)

Baby theories = Lawvere theories

Let S be a set. Then an S-**sorted set** is a disjoint union of sets indexed by S:

$$X = \coprod_{s \in S} X_s.$$

In other words, it is a function $X \to S$.

Lawvere's observation

Let $\mathbb{T}$ be a "baby" theory (i.e. with no type-dependency) with a set S of types.

Then every $\mathbb{T}$-model has an underlying S-sorted set.

E.g. the theory of ring-module pairs: every ring-module pair (A, M) has an underlying $\{a, m\}$-sorted set (i.e. *pair* of sets).

Lawvere's "recognition theorem"

Lawvere (1963) gave an **algebraic** description of what a "baby" theory (i.e. no type-dependency) is.

Lawvere's theorem

A baby theory (a.k.a. ***multisorted Lawvere theory****) is the data of*

1. *a set S of "types" or "sorts"*
2. *and a finitary monad on $[S, \mathcal{S}\mathrm{et}] = \mathcal{S}\mathrm{et}/S$.*

Recall

A **finitary** monad is one whose underlying endofunctor preserves filtered colimits.

Our goal: A "nice" definition of **dependently typed theory**

We want to give an algebraic description *à la Lawvere* of "adult" dependently typed theories.

This description should (obviously) strictly generalise Lawvere's.

Problem

All current definitions are *syntactic*, and it is not obvious to translate them into an algebraic description.

Some well-known syntactic definitions of what such a theory should be are GATs [Car78], FOLDS signatures [Pal16] and quotient inductive-inductive types (QIITs) [ACD$^+$18].

“Recognition theorem” for dependently typed theories

Our result (L.S., LeFanu Lumsdaine)

A **dependently typed theory** is the data of

1. a finitely branching inverse category I
2. and a finitary monad on $[I, \mathcal{S}\mathrm{et}]$.

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- Every Reedy category has a (wide non-full) inverse subcategory (e.g. Δ_+^{op})
- $$\begin{array}{c} G_1 \\ s\downdownarrows t \\ G_0 \end{array} \qquad \mathbb{G}^{\mathrm{op}} = \begin{array}{c} \vdots \\ s\downdownarrows t \\ G_2 \\ s\downdownarrows t \\ G_1 \\ s\downdownarrows t \\ G_0 \end{array} \qquad \mathbb{O}^{\mathrm{op}} \text{ (opetopes).}$$

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- The free-*weak*-ω-category monad on $\mathcal{S}\mathrm{et}^{\mathbb{G}^{\mathrm{op}}}$.
- For $T : \mathcal{S}\mathrm{et}^I \to \mathcal{S}\mathrm{et}^I$ a finitary cartesian monad, every T-operad (à la Burroni-Leinster).
- And many more...

Syntactic example

Let I be the category

$$G_2 \overset{s}{\underset{t}{\rightrightarrows}} G_1 \overset{s}{\underset{t}{\rightrightarrows}} G_0$$

with $s \circ s = s \circ t$ and $t \circ s = t \circ t$.

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Then I corresponds to the following type signature.

$$\vdash G_0 \qquad x, y : G_0 \vdash G_1(x, y) \qquad x, y : G_0, f, g : G_1(x, y) \vdash G_2(f, g)$$

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The theory of 2-categories is a theory with this type signature.

Preliminaries

- Let I be a small category.

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- Recall that $\mathrm{Fin}(I)$ is the finite-colimit completion of I^{op}. When I is a set, $\mathrm{Fin}(I)$ is the also the finite-coproduct completion of I.

Cartesian collections

The presheaf category

$$\mathcal{C}\mathrm{oll}_I := \mathcal{S}\mathrm{et}^{I \times \mathrm{Fin}(I)}$$

is called the category of I-**collections**.

(Intuition: $F \in \mathcal{C}\mathrm{oll}_I$ should be thought of as a *term signature* — for each *context* $\Gamma \in \mathrm{Fin}(I)$ and each *sort* $i \in I$, $F(i, \Gamma)$ is the set of operations with input Γ and output sort i.)

Composition of cartesian collections

I-collections can be composed via **substitution**:

$$G \circ F(i, \Gamma) := \int^{\Theta \in \mathrm{Fin}(I)} G(i, \Theta) \times \mathcal{S}\mathrm{et}^I(\Theta, F(-, \Gamma)).$$

$(\mathcal{C}\mathrm{oll}_I, \circ, E)$ is a (non-symmetric) **monoidal category**, where $E : \mathrm{Fin}(I) \hookrightarrow \mathcal{S}\mathrm{et}^I$.

Cartesian collections and endofunctors on $\mathcal{S}\mathrm{et}^I$

The functor $\mathrm{Lan}_E(-) : \mathcal{C}\mathrm{oll}_I \to [\mathcal{S}\mathrm{et}^I, \mathcal{S}\mathrm{et}^I]$ of left Kan extension along $E : \mathrm{Fin}(I) \hookrightarrow \mathcal{S}\mathrm{et}^I$ is (1) **fully faithful** and (2) **monoidal**.

$$\begin{array}{ccc} \mathrm{Fin}(I) & \xrightarrow{\;F\;} & \mathcal{S}\mathrm{et}^I \\ {\scriptstyle E}\big\downarrow & \cong & \nearrow {\scriptstyle \mathrm{Lan}_E F} \\ \mathcal{S}\mathrm{et}^I & & \end{array} \tag{1}$$

$$\mathrm{Lan}_E(F \circ G) \;\cong\; \mathrm{Lan}_E F \circ \mathrm{Lan}_E G \quad ; \quad \mathrm{Lan}_E E \;\cong\; \mathrm{id} \tag{2}$$

Consequence

$$\mathrm{Lan}_E - : \mathcal{M}\mathrm{on}(\mathcal{C}\mathrm{oll}_I, \circ, E) \hookrightarrow \mathcal{M}\mathrm{nd}(\mathcal{S}\mathrm{et}^I)$$

The category of monoids in $\mathcal{C}\mathrm{oll}_I$ is a full subcategory of the category of monads on $\mathcal{S}\mathrm{et}^I$. It is none other than the category of **finitary monads** on $\mathcal{S}\mathrm{et}^I$.

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Remarks

- We have only used that I is a small category.
- $\mathcal{M}\mathrm{on}(\mathcal{C}\mathrm{oll}_I, \circ, E)$is also known as the category of *monads with arities* (Weber) or *Lawvere theories with arities* (Melliès) for the arities $E : \mathrm{Fin}(I) \hookrightarrow \mathcal{S}\mathrm{et}^I$.

Contextual categories as monoids in collections

Inverse categories

Definition

An **inverse category** is:

- a small category I,
- whose objects are graded by "dimension" $dim : \mathrm{Ob}(I) \to \mathcal{O}\mathrm{rd}$,
- such that non-identity morphisms strictly decrease dimension,
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I is **finitely branching** if the tree $i/_I$ generated by every $i \in I$ is finite.

Main observation

Proposition (L.S., LeFanu Lumsdaine)

Let I be a finitely branching inverse category. Then $\mathrm{Fin}(I)^{\mathrm{op}}$ is equivalent to a contextual category $C(I)$ (the free contextual category on I).

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- Particular case: I is a set, then $\mathrm{Fin}(I)^{\mathrm{op}}$ is the free finite-product category on I.

Proof:

1. Note that:

- The Yoneda embedding factors as $\mathbf{y} : I^{\mathrm{op}} \hookrightarrow \mathrm{Fin}(I) \hookrightarrow \mathcal{S}\mathrm{et}^{I}$.

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- Every finite cell complex in $\mathcal{S}\mathrm{et}^I$ is finitely presentable.

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   - The boundary inclusions $\partial i \hookrightarrow \mathbf{y}i$ are finitely presentable (since i/I is finite).
   - Every finite cell complex in $\mathcal{S}\mathrm{et}^{I}$ is finitely presentable.
   - Every $X \in \mathrm{Fin}(I)$ can be written as a finite cell complex.

2. Define a **cell context** to be a finite sequence

$$\emptyset \to X_1 \to X_2 \to \ldots \to X$$

of *chosen* pushouts of boundary inclusions:

$$\begin{array}{ccc} \partial i & \longrightarrow & X_n \\ \big\downarrow & & \big\downarrow \\ \mathbf{y}i & \longrightarrow & X_{n+1}. \end{array}$$

Definition

The category Cell_I has as objects the cell contexts and as morphisms, $\mathrm{Cell}_I(\emptyset \ldots \to X, \emptyset \ldots \to Y) := \mathcal{S}\mathrm{et}^I(X, Y)$.

Clearly, $\mathrm{Cell}_I \simeq \mathrm{Fin}(I)$.

3. Not hard to see that $C(I) := \mathrm{Cell}_I^{\mathrm{op}}$ is a contextual category. (In fact, it is the *free contextual category on* I.) □

Remarks

- A collection $X \in \mathcal{C}\mathrm{oll}_I \simeq \mathcal{S}\mathrm{et}^{I \times \mathrm{Cell}_I}$ is now *literally* an I-sorted term signature.
- $C(I)$ has all finite limits.

I-contextual categories

Definition

An I-**contextual category** is a morphism of contextual categories $F : C(I) \to D$ such that in the (identity-on-objects, fully faithful) factorisation

$$C(I) \xrightarrow{F_1} D_I \xhookrightarrow{F_2} D, \qquad F = F_2 \circ F_1 : C(I) \to D$$

$F_2 : D_I \hookrightarrow D$ exhibits D as the *contextual completion* of D_I.

A morphism of I-contextual categories is a morphism in the coslice $C(I)/\mathcal{C}\mathrm{xlCat}$.

Theorem (L.S., LeFanu Lumsdaine)

The following categories are equivalent:

1. *The category* $\mathcal{C}\mathrm{xlCat}(I)$ *of* I*-contextual categories.*
2. *The category* $\mathcal{M}\mathrm{on}(\mathcal{C}\mathrm{oll}_I, \circ, E)$ *of monoids in* I*-sorted cartesian collections.*
3. *The category of finitary monads on* $\mathcal{S}\mathrm{et}^I$*.*

Proof.

Make use of the theory of Lawvere theories with arities [Mel10], [BMW12]. □

Summary, current and future work

- We introduce I-**contextual categories** as algebraic objects (monoids in collections) with an underlying dependently typed theory.
- We are working on a *linear* variant of this, and hoping to get a definition of *dependently coloured symmetric operad/linear dependently typed theory*.
- The "base change" properties of I-contextual categories remain to be understood.
- We would eventually like to add Id-types to this formalism.

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