Dependently typed theories as generalised Lawvere theories

Chaitanya Leena Subramaniam

j.w.w. Peter LeFanu Lumsdaine

IRIF, Université Paris Diderot

Dept. of Mathematics, Stockholms Universitet

Memorial conference for Erik Palmgren

19 Nov. 2020

Multisorted algebraic theories

An **algebraic theory** (or **Lawvere theory**) consists of :

1. A set S of sorts,
2. a set $\mathcal{F}$ of sorted function symbols, each written

$$x{:}A_1, \ldots x_n{:}A_n \vdash f : A \qquad (A_1, \ldots, A_n, A \in S).$$

3. A set of equations between terms over $\mathcal{F}$.

Dependently sorted algebraic theories

Question

What should a **dependently** sorted algebraic theory be?

Many approaches

- Cartmell's generalised algebraic theories [Car78].
- Makkai's logic with dependent sorts [Mak95].
- Fiore's Σ_n-models with substitution [Fio08].
- Palmgren's DFOL signatures [Pal16].
- Others (Aczel, Belo, QIITs ...)

I will present a strictly less general definition than all of the above

I will present a strictly less general definition than all of the above

. . . but which has the « *bon goût* » of closely resembling algebraic/Lawvere theories from a category-theoretic point of view.

I will present a strictly less general definition than all of the above

. . . but which has the « *bon goût* » of closely resembling algebraic/Lawvere theories from a category-theoretic point of view.

In fact, what I'll call **dependently typed theories** are exactly the Σ_0-models with substitution of [Fio08]. (I wish I had known this earlier.)

Equivalent categorical definitions of algebraic theory

Let $S \in \mathcal{S}\mathrm{et}$.

Definition

An S-sorted algebraic theory is a category with finite products whose objects are freely generated by S.

Definition

An S-sorted algebraic theory is a finitary monad on $\mathcal{S}\mathrm{et}/S = \widehat{S}$.

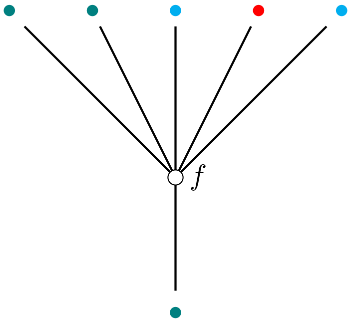
Let $\mathcal{F}\mathrm{in}(S) = \mathcal{F}\mathrm{in}/S$ be the (small) category of finite sets over S. Then the presheaf category $\mathcal{S}\mathrm{et}^{\mathcal{F}\mathrm{in}(S)\times S}$ of **cartesian S-sorted term signatures** has a "substitution" monoidal product.

Definition

An S-sorted algebraic theory is a monoid in the monoidal category $\mathcal{S}\mathrm{et}^{\mathcal{F}\mathrm{in}(S)\times S}$.

Combinatorics

These definitions are based on a combinatorial view of substitution of sorted terms of an algebraic theory as “cartesian” grafting of trees (cartesian = with weakening and duplication of inputs).



A term in a multisorted Lawvere theory takes a finite coproduct of sorts as input, and has an output sort.

Whence S-sorted term signatures as objects of $\mathcal{S}\mathrm{et}^{\mathcal{F}\mathrm{in}(S)\times S}$.

Generalising this picture to dependent types

Dependent type signatures

We begin with a syntactic definition.

A **(dependent type) signature** S is a graded set $S = \coprod_{n \in \mathbb{N}} S_n$, such that each S_j is a set of *type declarations* over the signature $S_{<j} = \coprod_{i<j} S_i$.

A **type declaration** over a signature S is a pair (Γ, A) where Γ is a (Martin-Löf) context typed by S and A is a (fresh) type symbol.

Examples

- $$\vdash \mathrm{Ob}$$
  $$x, y{:}\mathrm{Ob} \vdash \mathrm{Hom}(x, y)$$

- $$\vdash C$$
  $$\forall k \in \mathbb{N}, \quad x_1, \ldots, x_k, y{:}C \vdash O(x_1, \ldots, x_k, y)$$

Every signature S has a **syntactic category** whose objects are contexts Γ typed by S and whose morphisms are context morphisms $\Gamma \to \Delta$. Since there are are no term symbols, all morphisms are substitutions of variables only.

For a signature S, let C_S be the full subcategory of its category of contexts on the contexts $(\Gamma, x{:}A)$ for all (Γ, A) in S.

Categorical parenthesis

Definition

A **direct** category is a small category C such that the relation

$$c < d \qquad \Leftrightarrow \qquad \exists \text{ a non-identity arrow } c \to d$$

on the objects of C is well-founded (i.e. no infinite chains $\ldots < c_0$).

Definition

A category C is **locally finite** if each of its slice categories is finite.

Categorical definition of a signature

Signatures are precisely locally finite direct categories (cf. [Mak95, Fio08]).

Proposition

The map $S \mapsto C_S^{\mathrm{op}}$ is an equivalence between signatures and locally finite direct categories.

Examples of dependent type signatures

0. Any set S, seen as a discrete category.
1. The category $\{s, t : 0 \rightrightarrows 1\}$.
2. The category $\mathbb{G}$ of globes.
3. The category Δ_+ of semi-simplices.
4. The category $\mathrm{elt}_{\mathrm{pl}}$ of planar corollas/elementary trees.
5. The category Ω_{pl} of planar trees.
6. The category $\mathbb{O}$ of opetopes.

Dependently typed theories

Syntactically, a **dependently typed theory** consists of:

1. A dependent type signature S,
2. an ordered set of *term declarations* of the form $\Gamma \vdash f : A\sigma$,
3. and an ordered set of equations of the form $\Gamma \vdash t_1 = t_2 : A\sigma$,

where Γ is any context typed by S, σ is a term substitution, and t_1 and t_2 are typed terms.

Example of a dependently typed theory

The theory of **categories**:

$$\begin{aligned} &\vdash \mathrm{Ob} \\ x, y{:}\mathrm{Ob} &\vdash \mathrm{Hom}(x, y) \\ x{:}\mathrm{Ob} &\vdash 1_x : \mathrm{Hom}(x, x) \\ \ldots, a{:}\mathrm{Hom}(x, y), b{:}\mathrm{Hom}(y, z) &\vdash b \circ a : \mathrm{Hom}(x, z) \\ x, y{:}\mathrm{Ob}, a{:}\mathrm{Hom}(x, y) &\vdash a \circ 1_x = a : \mathrm{Hom}(x, y) \\ x, y{:}\mathrm{Ob}, a{:}\mathrm{Hom}(x, y) &\vdash 1_y \circ a = a : \mathrm{Hom}(x, y) \\ \ldots &\vdash (c \circ b) \circ a = c \circ (b \circ a) : \mathrm{Hom}(x_1, x_4) \end{aligned}$$

We will see that there are dependently typed theories of 2-categories, n-categories, ω-categories, reflexive graphs, simplicial sets, opetopic sets, planar operads . . .

Categorical definition

Let $\mathbb{S}$ be a locally finite direct category (let S be the corresponding signature). Let $\mathcal{F}\mathrm{in}(\mathbb{S})$ denote the full subcategory of $\widehat{\mathbb{S}}$ of the finitely presentable objects.

Recall that any X in $\widehat{\mathbb{S}}$ is in $\mathcal{F}\mathrm{in}(\mathbb{S})$ just when X is a finite colimit of representables.

For each s in $\mathbb{S}$, let $\mathbb{S}^-_{/s}$ denote the full subcategory of the slice category $\mathbb{S}/s$ such that the only object not in $\mathbb{S}^-_{/s}$ is the identity morphism $1_s : s \to s$. The colimit of the functor $\mathbb{S}^-_{/s} \to \mathbb{S} \hookrightarrow \widehat{\mathbb{S}}$ is a subobject $\partial s \hookrightarrow s$ called the **boundary** of the representable presheaf s.

Since $\mathbb{S}$ is locally finite, ∂s is finitely presentable for every s in $\mathbb{S}$.

Definition

A **finite cell complex** is a finite sequence of morphisms $\emptyset \to X_0 \ldots \to X_n$ in $\widehat{\mathbb{S}}$ where each morphism $X_i \to X_{i+1}$ is a *chosen* pushout of some $\partial s \hookrightarrow s$.

Lemma

Any X in $\widehat{\mathbb{S}}$ is finitely presentable if and only if there exists a finite cell complex $\emptyset \to \ldots \to X$.

We define $\mathcal{C}\mathrm{ell}(\mathbb{S})$ to be the category whose objects are finite cell complexes, and such that $\mathrm{Hom}(\emptyset \ldots \to X, \emptyset \ldots \to Y) = \widehat{\mathbb{S}}(X, Y)$.

Clearly, the functor $\mathcal{C}\mathrm{ell}(\mathbb{S}) \to \mathcal{F}\mathrm{in}(\mathbb{S})$ is an equivalence of categories.

Proposition

$\mathcal{C}\mathrm{ell}(\mathbb{S})^{\mathrm{op}}$ is *isomorphic* to the syntactic category of the signature S.

Corollary

$\mathcal{C}\mathrm{ell}(\mathbb{S})^{\mathrm{op}}$ is a contextual category.

Definition

The category $\mathcal{C}\mathrm{oll}_{\mathbb{S}}$ of **cartesian $\mathbb{S}$-sorted term signatures** is defined to be the presheaf category $\left[\mathcal{C}\mathrm{ell}(\mathbb{S}), \widehat{\mathbb{S}}\right]$.

For every context Γ typed by S and every type declaration s in S, a term signature X in $\mathcal{C}\mathrm{oll}_{\mathbb{S}}$ gives (functorially) a set $(X\Gamma)_s$ of *term declarations* of type s in the context Γ.

Categorical parenthesis

Let C be a small category and let $\mathcal{F}\mathrm{in}(C)$ be as previously.

The presheaf category $\left[\mathcal{F}\mathrm{in}(C), \widehat{C}\right]$ has a "substitution" monoidal product defined by

$$((Y \circ X)\Gamma)_c := \int^{\Theta \in \mathcal{F}\mathrm{in}(C)} (Y\Theta)_c \times \widehat{C}(\Theta, X\Gamma)$$

whose unit is the inclusion functor $E : \mathcal{F}\mathrm{in}(C) \hookrightarrow \widehat{C}$.

The functor $\mathrm{Lan}_E(-) : \left[\mathcal{F}\mathrm{in}(C), \widehat{C}\right] \to \left[\widehat{C}, \widehat{C}\right]$ of left Kan extension along $E : \mathcal{F}\mathrm{in}(C) \hookrightarrow \widehat{C}$ is (1) **fully faithful** and (2) **monoidal**.

$$\begin{array}{ccc} \mathcal{F}\mathrm{in}(C) & \xrightarrow{\;X\;} & \widehat{C} \\ {\scriptstyle E}\big\downarrow & \cong & \nearrow_{\mathrm{Lan}_E X} \\ \widehat{C} & & \end{array} \qquad (1)$$

$$\mathrm{Lan}_E(Y \circ X) \;\cong\; \mathrm{Lan}_E Y \circ \mathrm{Lan}_E X \quad ; \quad \mathrm{Lan}_E E \;\cong\; \mathrm{id}_{\widehat{C}} \qquad (2)$$

Proposition

There is an equivalence of categories between monoids in $\left[\mathcal{F}\mathrm{in}(C), \widehat{C}\right]$ and finitary monads on $\widehat{C}$.

Equivalent definitions of dependently typed theory

From the previous parenthesis, we have a substitution monoidal product on $\mathcal{C}\mathrm{oll}_{\mathbb{S}}$.

The **term algebra** of $X \in \mathcal{C}\mathrm{oll}_{\mathbb{S}}$ is the free monoid on X.

Definition

An $\mathbb{S}$-sorted **dependently typed theory** is a monoid in $\mathcal{C}\mathrm{oll}_{\mathbb{S}} \simeq \left[\mathcal{F}\mathrm{in}(\mathbb{S}), \widehat{\mathbb{S}}\right]$.

Definition

An $\mathbb{S}$-sorted **dependently typed theory** is a finitary monad on $\widehat{\mathbb{S}}$.

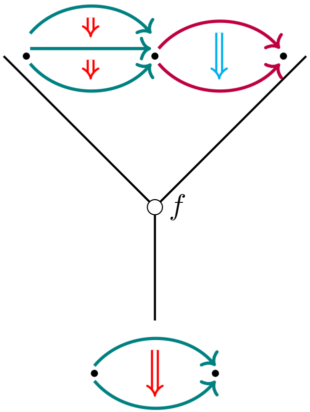
Definition

An $\mathbb{S}$-sorted **dependently typed theory** is an $\mathbb{S}$-contextual category.

The last definition generalises the “finite-product category” definition of algebraic theories.

Combinatorics of dependently typed theories

The substitution monoidal product for $\mathbb{S}$-sorted terms can also be seen as "cartesian" grafting of trees.



A term of an $\mathbb{S}$-sorted dependently typed theory takes a finite cell complex as input, and has as output sort a cell (i.e. an object of $\mathbb{S}$).

(This point of view is closely related to Burroni-Leinster T-operads.)

Examples of dependently typed theories

0. Every multisorted algebraic theory.
1. The identity monads on $\mathcal{G}\mathrm{ph}, \widehat{\mathbb{G}}, \widehat{\Delta_+}, \widehat{\mathrm{elt}_{\mathrm{pl}}}, \widehat{\Omega_{\mathrm{pl}}}, \widehat{\mathbb{O}}$.
2. The free-category monad on $\mathcal{G}\mathrm{ph}$.
3. The free-planar (coloured) operad monad on $\widehat{\mathrm{elt}_{\mathrm{pl}}}$.
4. The free simplicial set monad on semi-simplicial sets.
5. The free-strict-ω-category monad on $\widehat{\mathbb{G}}$.
6. The free-weak-ω-category monad on $\widehat{\mathbb{G}}$.
7. For $T : \widehat{\mathbb{S}} \to \widehat{\mathbb{S}}$ a finitary cartesian monad, every T-operad (à la Burroni-Leinster).

Theorem (L.S., LeFanu Lumsdaine)

The following categories are equivalent:

1. *The category* $\mathcal{C}\mathrm{xlCat}(\mathbb{S})$ *of* $\mathbb{S}$*-contextual categories.*
2. *The category* $\mathcal{M}\mathrm{on}(\mathcal{C}\mathrm{oll}_{\mathbb{S}}, \circ, E)$ *of monoids in cartesian* $\mathbb{S}$*-sorted term signatures.*
3. *The category of finitary monads on* $\widehat{\mathbb{S}}$*.*

3'. *The category of Lawvere theories with arities* $\mathcal{C}\mathrm{ell}(\mathbb{S}) \to \widehat{\mathbb{S}}$*.*

Conclusion

In sum,

- We introduce **dependently typed theories** as a generalisation of multisorted algebraic theories.
- These "cartesian dependent multicategories" are less expressive than many other syntactic approaches, but have a nice algebraic description.
- They manage to capture a large number of well-known examples.

Reflections on dependently coloured operads

Regular algebraic theories

A term $\Gamma \vdash t : A$ of a multisorted algebraic theory is **linear** (or "planar") if each variable in Γ appears exactly once in t, and in the same order as in Γ.

A multisorted algebraic theory is **strongly regular** if each of its equations is between "linear" terms.

Strongly regular algebraic theories and planar coloured operads are closely related.

Coloured operads

Let S be a set of sorts ("colours" in operad jargon). Then the free monoidal category on S is (equivalent to) the set ΣS of finite lists of elements of S. There is an obvious surjective on objects functor $\Sigma S \to \mathcal{F}\mathrm{in}(S)$ taking $(s_1, \ldots, s_k)$ to the coproduct of the representables $s_1, \ldots, s_k$.

The category of **linear** S-sorted term signatures is the presheaf category $\mathcal{S}\mathrm{et}/(\Sigma S \times S) = \left[\Sigma S, \widehat{S}\right]$.

The linear substitution monoidal product on $\left[\Sigma S, \widehat{S}\right]$ is given by **convolution** :

First, for $X \in \left[\Sigma S, \widehat{S}\right]$ and $(s_1, \ldots, s_k) \in \Sigma S$ we define $X^{(s_1,\ldots,s_k)} \in [\Sigma S, \mathcal{S}\mathrm{et}]$ as the Day convolution

$$(X_{s_1} \otimes \ldots \otimes X_{s_k})_{(s'_1,\ldots,s'_m)} := \sum_{\substack{f:\{s_1,\ldots,s_k\}\to\Sigma S \\ fs_1+\ldots+fs_k=(s'_1,\ldots,s'_m)}} \prod_{i=1}^{k} X(fs_i)_{s_i}.$$

$(X^{(s_1,\ldots,s_k)})_{(s'_1,\ldots,s'_m)}$ is the set of *linear* substitutions $(s'_1, \ldots, s'_m) \to (s_1 \ldots, s_k)$ using terms from X.

Next, for $X, Y \in \left[\Sigma S, \widehat{S}\right]$, we define $(Y \circ X) \in \left[\Sigma S, \widehat{S}\right]$ by

$$((Y \circ X)\bar{v})_s := \sum_{\bar{w} \in \Sigma S} (Y\bar{w})_s \times (X^{\bar{w}})_{\bar{v}}$$

This is just the combinatorics of grafting planar labeled trees.

An S-coloured planar operad is a monoid in $\left[\Sigma S, \widehat{S}\right]$.

“Convolution” for $\mathbb{S}$-sorted signatures?

For a dependent type signature $\mathbb{S}$, there seems to be an analogous category $\Sigma\mathbb{S}$ with essentially the same objects as $\mathcal{C}\mathrm{ell}(\mathbb{S})$.

Given an object $X \in \left[\Sigma\mathbb{S}, \widehat{\mathbb{S}}\right]$ and $\Gamma, \Delta \in \mathcal{C}\mathrm{ell}(\mathbb{S})$, we can define the set of linear substitutions $\Delta \to \Gamma$ using terms from X as

$$\sum_{\substack{f:\mathbb{S}/\Gamma \to \Sigma\mathbb{S} \\ \text{“colim”} f = \Delta}} \int_{x:s \to \Gamma} X(fx)_s,$$

where the end is over the functor

$$\mathbb{S}/\Gamma \times (\mathbb{S}/\Gamma)^{\mathrm{op}} \xrightarrow{f \times \mathsf{p}} \mathcal{C}\mathrm{ell}(\mathbb{S}) \times \mathbb{S}^{\mathrm{op}} \xrightarrow{X} \mathcal{S}\mathrm{et}.$$

Question

Does this give a monoidal product and a notion of dependently coloured operad?

JW Cartmell.
Generalised algebraic theories and contextual categories.
PhD thesis, University of Oxford, 1978.

Marcelo Fiore.
Second-order and dependently-sorted abstract syntax.
In *LICS 2008*, pages 57–68. IEEE, 2008.

Michael Makkai.
First order logic with dependent sorts, with applications to category theory.
1995.

Erik Palmgren.
Categories with families, FOLDS and logic enriched type theory.
arXiv preprint arXiv:1605.01586, 2016.