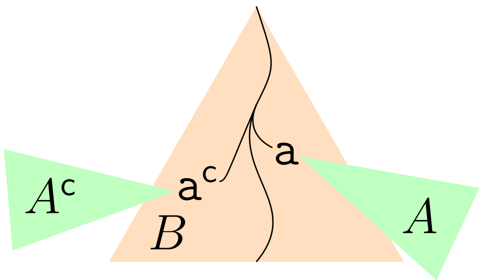
Bachelor Project – Spring 2021

Infinite games in the Baire space

With 34 Illustrations



Author
Diego Dorn

EPFL

Supervisor
Jacques Duparc

Introduction

Descriptive set theory is the study of subsets of the reals and other completely metrisable separable spaces, called Polish spaces. It classifies subsets of the reals of increasing complexity into hierarchies, like the Borel hierarchy, that captures complexity through the minimum amount of operations of countable unions and complementation needed to construct a set, starting with the open sets. An other, much finer, hierarchy is Wadge's hierarchy that captures complexity through reduction by continuous functions.

Baire noticed[1] that the space of sequences of integers endowed with the prefix topology, later called the *Baire space* by Bourbaki, has many useful properties and those will ease our study of the aforementioned hierarchies.

Our main tool for the analysis of those hierarchies and spaces will be infinite games of perfect information between two players. We will construct games such that one of the player has a winning strategy if and only if some function or set satisfy a given property, such as a function being continuous.

Acknowledgments

First and foremost, I would like to thank my supervisor, Jacques Duparc for his guidance during my exploration of descriptive set theory and infinite games. He showed me many interesting results, a lot of which I did not reproduce here, and took his time to explain their proofs (sometimes more than once!) in a very visual way, sharing a lot of intuition. Thank you also, for your teaching style and enthusiasm in the classes of *Mathematical Logic* and *Set Theory*, which have greatly amplified my appetite for those subjects and filled me with wonder ever Wednesday.

I would also like to thank my friend Gabin Kolly, for all the interesting mathematical discussions that we had, for always stimulating my curiosity, challenging my intuition with surprising counter-examples and spotting the good ideas in the flow of non-sense I produce while trying to figure things out.

Lastly, I wish to thank Michael Paper for sparking my interest in logic and informatics and to always support me when I need it. I will never be grateful enough to my parents and family, who always make everything they can to help me turn my dreams into reality. It is a blessing to have a family that always support me in my choices, even when I choose work over a precious moment together.

Contents

In the first section, we setup the context of trees and polish spaces, while proving simple results that we will use later. We also replace the Baire space in the context of real analysis by showing that it is isomorphic to the irrationals.

In the second section, we introduce four hierarchies: the Borel hierarchy, the Wadge hierarchy, and two hierarchies of function based on the Borel hierarchy. We explicit and

[1] See for instance [Baire, 1899] for the inception of the *théorie des ensembles de suites d'entiers*.

prove the shape of the Borel hierarchy and introduce shortly the analytic sets, that lay above the Borel sets.

We then introduce infinite games between two players in the third section, and most importantly Wadge's game, that will allow us to study the Wadge hierarchy. We also discuss strategies and the determinacy of such games, that is, whether there always is a winning strategy for one of the players.

In the fourth section, we familiarize ourselves with infinite games and construct games where the second player has a winning strategy if and only if a function is continuous, and similarly for what we'll call $\Lambda_{2,2}$ function. We then introduce the tree game of Semmes, and our main theorem is that a function is Borel if and only if the second player has a winning strategy in this game.

In the last section, we look constructing sets of the first $\omega_1^{\omega_1}$ levels of Wadge's hierarchy. To that extent, we introduce to the new concept of *automatic set* as a way to build new sets of a given rank. Finally, we also prove a theorem from Hausdorff for a different point of view on the first ω_1 levels, through the difference hierarchy.

Contents

1 The Baire Space

Throughout this document, we adopt a set theoretic point of view and use integers with the von Neumann construction, that is integers are ordinals and $n = \{0, 1, \dots, n-1\}$. We write ω for the set of the integers and the first infinite ordinal, we use ω_1 for the first uncountable ordinal.

For conciseness, we may use the symbol $\exists^\infty$ in a formula such as $\exists^\infty x \in X P(x)$ to say that there exists an infinity of x that satisfy $P(x)$. For instance if s is a sequence, we may write "$\exists^\infty n \in \omega\ (s_n = 0)$" to say that there is an infinite number of zeros in the sequence. Note that the quantifier is equivalent to $\forall N \in \omega\ \exists n > N$.

Regarding the inclusion of sets, we write $A \subset B$ to say that A is contained in B, not necessarily strictly and we use $A \subsetneq B$ when A is properly contained in B. We will write $A^\mathsf{c} = X \setminus A$ for the complement of A in a set X, whenever the set X is clear from the context. Given a set X, we write $\mathcal{P}(X)$ for the set of subsets of X.

We will use **ZF** in boldface for Zermelo-Frankel's set theory from the first order logic, **AC** for the axiom of choice and **ZFC** = **ZF** $\cup \{AC\}$ for set theory with the axiom of choice. Most of our proofs will however not take place in **ZFC**, but in **ZF** + {**DC**}, where **DC** is the axiom of dependent choice. Indeed, we will not need more choice than **DC**.

1.1 Trees and sequences

We start by recalling important notions and definitions about sequences and trees that we will use often. Most of those are classical, be we will also use a few new notations to make our work simpler and clearer.

Definition 1.1 (Sequences)**.** Let α be an ordinal and X a set. A **α-sequence** s is a function from α to X:

$$s : \alpha \to X$$

The elements of a sequence are then $x(0), x(1), \dots$ but we may write them as $x_0, x_1, \dots$ as it is less verbose.

A **finite sequence** is a sequence from a finite ordinal. If s is a finite sequence, we can write it as a n-uplet $x = (x_0, \dots, x_{n-1})$.

If s is an α-sequence, we call α the **length** of the sequence. We write

$$\mathrm{lh}(s) := \alpha.$$

Most of the time, we will work with finite sequences or ω-sequences, so when we use the term sequence without specifying the ordinal we will use the convention that it is an ω-sequence.

Notation 1.2. Let X be any set. We write

- ε for the empty sequence, so $\mathrm{lh}(\varepsilon) = 0$.

- $X^{<\omega}$ for the set of finite sequences on X.
- X^{ω} for the set of all ω-sequences.
- $X^{\leq\omega} = X^{<\omega} \cup X^{\omega}$ the set of sequences of length lower or equal to ω.
- If s is an α-sequence and $\beta < \alpha$ an ordinal, we write $s\restriction_{\beta}$ for the restriction of s to β.
- For $s \in X^{\leq\omega}$, we use $\mathrm{even}(s) = (s_0, s_2, \dots)$ for the even terms of the sequence, and $\mathrm{odd}(s) = (s_1, s_3, \dots)$ for the sequence of odd terms.
- For $s \in X^{<\omega} \setminus \{\varepsilon\}$, we use $\mathrm{last}(s) = s(\mathrm{lh}(s) - 1)$ for the last element of the sequence.
- For two sequences s, u, we write $s \subset u$ when s is a prefix of u, i.e. $u\restriction_{\mathrm{lh}(s)} = s$. This makes sense when we consider the set-theoretic representation of functions, which are described by their graphs.

Definition 1.3 (Trees)**.** A **tree** on a set X is a set $T \subset X^{<\omega}$ of finite sequences that is closed by prefix, so that for each sequence $s \in T$, and each $n < \mathrm{lh}(s)$, the restriction $s\restriction_n \in T$.

A **successor** of $s \in T$ is a sequence $t \in T$ such that $\mathrm{lh}(t) = \mathrm{lh}(s) + 1$ and $t\restriction_{\mathrm{lh}(s)} = s$. We call $\mathrm{succ}(s)$ the set of successors (or children) of s.

An **infinite branch** of T is a sequence $s \in X^{\omega}$ such that for all $n \in \mathbb{N}$, the prefix $s\restriction_n$ belongs to T.

A sequence $s \in T$ is a **terminal node** if it is not the prefix of a longer sequence. A tree without terminal nodes is called **pruned**.

Definition 1.4. A **colored tree** is a pair (T, c) where T is a tree and $c : T \to X$ is a coloring function with values in some set X.

The purpose of colored trees is to associate a value to each node of the tree, so that we can see it for instance as a labeled tree, with the labels containing any data of interest. Note that a colored tree is fully determined by its coloring function, since the tree is the domain of the coloring: $T = \mathrm{dom}(c)$.

1.2 Polish spaces

Definition 1.5. A topological space X is **separable** if there exist a countable set $S \subset X$ that is dense.

For instance, every countable space is separable, but also any space that has a countable basis, like $\mathbb{R}^n$, $\mathbb{C}^n$, or any manifold. Examples of spaces that do not have a countable basis but are still separable are $[0,1]^{[0,1]}$ and $\mathbb{R}^{\mathbb{R}}$ with the product topology, or $\mathbb{Q}\,/\,\mathbb{Z}$, $\mathbb{R}\,/\,\mathbb{Z}$.

Definition 1.6. A **metric** on X is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ that satisfies three properties:

- identity of indiscernibles: $d(x,y) = 0 \iff x = y$.
- reflexivity: $d(x,y) = d(y,x)$.
- triangle inequality: $d(x,y) + d(y,z) \geq d(x,z)$.

The metric is said to be **complete** if every Cauchy sequence converges.

Definition 1.7. A topological space X is **metrisable** if there exists a metric d that is compatible with the topology, that is, if the topology generated by all the balls

$$B_d(x_0, r) = \{x \in X \mid d(x, x_0) < r\} \quad \text{with} \quad x_0 \in X, r > 0$$

is the same as the topology in X. Furthermore, if one such metric is complete, we say that X is **completely metrisable**

Definition 1.8. A **polish space** is a topological space that is separable and completely metrisable.

Polish spaces are very natural and have a lot of properties. There are a lot of examples of polish spaces, but the most important ones are:

- $\mathbb{R}^n$, $\mathbb{C}^n$ and $[0,1]$ are separable and completely metrisable, hence polish, but spaces like $(0,1)$ even if not complete with the usual metric are still completely metrisable and thus polish. In fact, we will show that in a completely metrisable space, any countable intersection of open subsets is also completely metrisable.
- The **Baire space**, ω^ω which is the space of all sequences of integers with the initial segment topology. Its complete metric is defined by $d(x,y) = 2^{-n}$, where n is the first index where the two sequences are different. We will focus on the study of this space since many properties of other Polish spaces can be deduced from the Baire space.

  It also has the very useful property of being homeomorphic to its powers, finite or countable, which gives it a very practical advantage over $\mathbb{R}$.

- The **Cantor space** is the space of sequences of 0s and 1s. It is denoted $\mathcal{C}$ and can be alternatively defined as the set of reals in $[0,1]$ without any 1 in their base 3 expansion. It can also be seen as $2^\omega = \{0,1\}^\omega$ with the product topology.

- The **Hilbert Cube** is $[0,1]^\omega$ equipped with the product topology and a complete metric is defined in the same way as for the Baire space.

Proposition 1.9. Let X be a polish space. A set $A \subset X$ is a polish space if and only if there exists a sequence of open sets $(U_n)_{n\in\omega}$ such that

$$A = \bigcap_{n\in\omega} U_n.$$

Proof. We first show that if $U_n, n \in \mathbb{N}$ is a sequence of open sets, $A = \bigcup_n U_n$ is polish. Since X is metrisable and separable, it has a countable basis, and thus A has a countable basis and is also separable. Now let d be a complete metric on X. We can construct a complete metric d_n on U_n:

$$d_n(x,y) = d(x,y) + \left| \frac{1}{d(x, U_n^\mathsf{c})} - \frac{1}{d(y, U_n^\mathsf{c})|} \right|$$

where $d(x, U_n^\mathsf{c})$ is the distance between x and the complement of U_n. Then, we can construct a bounded complete metric, $\tilde{d}_n$, on U_n, such that for all $x, y \in U_n$, $\tilde{d}_n(x,y) \leq 1$

$$\tilde{d}_n(x,y) = \min\big(d_n(x,y), 1\big).$$

Finally, we can combine all those metrics into a complete metric on A, defined by

$$d_A(x,y) = \sum_{n\in\mathbb{N}} 2^{-n} \cdot \tilde{d}_n(x,y).$$

It is not hard and not very enlightening to show that all those metrics satisfy the desired properties.

For the other direction, assume that ρ is a complete metric on $A \subset X$. We need to show that A is a countable intersection of open sets. For a bounded set $U \subset A$, we recall that the diameter of U is

$$\operatorname{diam}_\rho(U) = \inf\{r \in \mathbb{R}_+ \mid \exists x \in A\ U \subset B_\rho(x,r)\}$$

We consider the set

$$B = \left\{x \in \overline{A} \,\middle|\, \inf \operatorname{diam}_\rho(A \cap U) = 0\right\},$$

where the union is taken over all (bounded) open neighbourhoods of x. B is the set of points that seem very close to A, according to the distance ρ. Now we notice that B is a countable union of open sets since $\overline{A}$ is a closed set and thus a countable intersection of open sets and

$$\begin{aligned} B &= \left\{x \in \overline{A} \,\middle|\, \inf \operatorname{diam}_\rho(A \cap U) = 0\right\} \\ &= \bigcap_{k\in\omega} \left\{x \in \overline{A} \,\middle|\, \inf \operatorname{diam}_\rho(A \cap U) < \frac{1}{k}\right\} \\ &= \overline{A} \cap \bigcap_{k\in\omega} \bigcup_{U} U \end{aligned}$$

where the last union is taken over all the open sets U such that $\operatorname{diam}_\rho(A \cap U) < \frac{1}{k}$. Now we have that $A \subset B$ since for any $x \in A$ the balls $B_\rho(x, 1/k)$ satisfy the conditions.

It remains to show that $B \subset A$ to conclude. To that extent, consider a point $x \in A$ and a sequence $(y_n)_{n\in\omega}$ in A converging to x, that is, given a neighbourhood of x, there exist a point starting which the sequence is contained in this neighbourhood. Since $x \in A$, we can use the neighbourhoods $A \cap U$ from the definition to see that the sequence (y_n) is Cauchy with respect to ρ and thus converges to a point in A. Thus $x \in A$ and $A = B$. □

1.3 The Baire Space

Definition 1.10. The **Baire space**, ω^ω, is space of all infinite sequences of integers. For $x \neq y \in X$, we set n to be the first position where the two sequences differ, then we define the metric as

$$d(x, y) = 2^{-n}.$$

Two points are thus close if they have a large common initial segment. The balls generated by this metric are simple, $B(x, 2^{-n})$ is precisely the set of all sequences that start with the same n integers as x.

Let s be a finite sequence, the **basic open set** generated by s is $[s] \subset \omega^\omega$, the set of all sequences that start with s.

The basic open sets $[s]$ are important, as they form the standard topological basis of ω^ω, and are both open and closed, indeed

$$[s] = \omega^\omega \setminus \bigcup [t]$$

where the union ranges over all the finite sequences t that do not share a non-empty prefix with s. Hence, the Baire space admits a basis of clopen sets and is totally disconnected.

The Baire space is always viewed with both as a metric space, but also as the infinite branches of the tree $\omega^{<\omega}$, and thinking about the tree structure can often give a good intuition.

The following proposition replaces the Baire space in a more analytic context by showing that it is isomorphic to the irrational numbers.

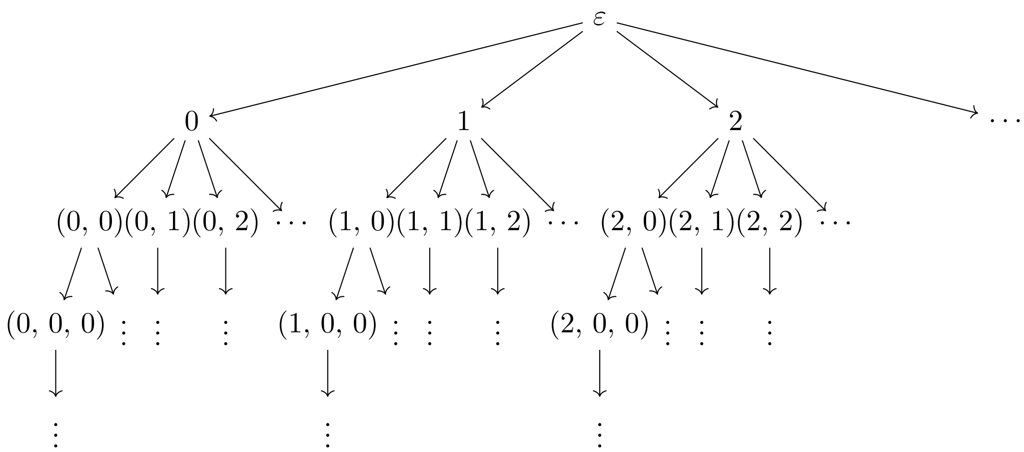


Figure 1: The Baire space as a tree

There exists mostly two proofs of this fact. The first one relies on the theory of continuous fractions, associating every sequence $(x_n) \in \omega^\omega$ with the irrational

$$x = x_0 + \frac{1}{1 + x_1 + \frac{1}{1+x_2+\frac{1}{1+x_3+\frac{1}{\cdots}}}},$$

however this proof involves more analytical tools and results from the study of continuous fractions, which are of great interest but outside the scope of this document.

We present here a simpler proof due to [Miller, 1995] that involves only elementary topological facts about $\mathbb{R}$.

Proposition 1.11. The Baire space is isomorphic to the irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$.

Proof. We first fix an enumeration, (q_n) of $\mathbb{Q}$ and we notice that the Baire space is isomorphic to $\mathbb{Z}^\omega$, just by applying any bijection between ω and $\mathbb{Z}$ to each element of the sequences.

The main idea is to recursively partition intervals in $\mathbb{R}$ into sub intervals, and each interval will be in correspondence with a basic open set in the Baire space, so that both have the same structure.

We construct a tree T of intervals such that, for each node $s \in \omega^{<\omega}$, T_s is an open interval $]a_s, b_s[$, with both a_s and b_s rational numbers.

- For the empty sequence, we set $T_\varepsilon = \mathbb{R}$.
- If $T_s =]a_s, b_s[$ and L is the length of s, the children of T_s are then $T_{s^\frown n} =]r_n, r_{n+1}[$ where $(r_n)_{n\in\mathbb{Z}}$ is a sequence of rationals such that
  - the sequence is strictly increasing.
  - the sequence *starts* at a_s, i.e. $\lim_{n\to-\infty} r_n = a_s$.

– the sequence *ends* as b_s, i.e. $\lim_{n\to\infty} r_n = b_s$.
– $|r_n - r_{n+1}| < 1/L$, the intervals are smaller when L is large.
– if $q_L \in T_s$, $r_0 = q_L$, i.e. q_L is selected at most on the level L.

Clearly the infinite branches of this tree are isomorphic to ω^ω, since every node has countably many children. We just need to show that they are also isomorphic to the irrationals.

Let s be an infinite branch of T, let

$$U = \bigcap_{n\in\mathbb{N}} T_{s\restriction_n} = \bigcap_{n\in\mathbb{N}}]a_{s\restriction_n}, b_{s\restriction_n}[$$

Since $|a_{s\restriction_n} - b_{s\restriction_n}| < 1/n$, and both sequences are monotonous by construction, we have $\lim a_{s\restriction_n} = \lim b_{s\restriction_n} = x$. Since this x is in each $T_{s\restriction_n}$, we have that $U = \{x\}$, and our map is given by $f : s \mapsto x$.

- It is irrational valued, as its range contains no rational, since any q_n is contained in no interval associated with a sequence of length greater that n.
- It is a bijection, with inverse $f^{-1}(x) = s$, where s is constructed by induction: $s_0 = T_\varepsilon$ and $s_{n+1} = k$ if $x \in T_{s\restriction_n \frown k}$. Such k always exists since every irrational of $T_{s\restriction_n}$ is contained in exactly on of the children.
- It is continuous since the open sets $]a, b[\setminus \mathbb{Q}$ with $a, b \in \mathbb{Q}$ form a basis of the topology on $\mathbb{R} \setminus \mathbb{Q}$ and each such set is the unions of some $T_s \setminus \mathbb{Q}$ so its preimage is the union of some open sets $[s]$.
- It is open since the image of any open ball $[s]$ is exactly the ball $T_s \setminus \mathbb{Q}$.

Thus we have constructed an isomorphism between ω^ω and $\mathbb{R} \setminus \mathbb{Q}$. □

2 Hierarchies of sets and functions

Having laid the foundations, we will now focus on building hierarchies whose interest is at the heart of descriptive set theory. The purpose of those hierarchies is to classify subsets of polish spaces using different notions of complexity.

2.1 Borel's hierarchy

Let X be any polish space, for instance $\mathbb{R}$ or ω^ω. The Baire hierarchy on X is a classification of the subsets of X, where the sets that are "simpler" are at the bottom of the hierarchy and get more complex as we go up the hierarchy. In this hierarchy, the "simplest" sets are the open and closed sets, and the complexity is measured in terms of the number of operations of *countable union* and *countable intersection* that are needed to attain a given set, starting with the open sets, that is, we consider that at the lowest level lies the open and closed sets, then countable unions of closed sets and countable intersections of open sets (since countable unions of open sets are still open sets, we do not consider them). Above that would stand the countable unions of countable intersections of closed sets... and likewise into the transfinite.

More formally, we define three classes, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$ by transfinite induction.

Definition 2.1 (Borel's hierarchy)**.** The Borel hierarchy on a topological space X is a collection of classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$ for each ordinal $\alpha > 0$, that we define by transfinite induction:

- $\mathbf{\Sigma}^0_1$ is the collection of the open sets of X.
- $\mathbf{\Pi}^0_\alpha$ is the collection of the complements of $\mathbf{\Sigma}^0_\alpha$, i.e.

$$A \in \mathbf{\Pi}^0_\alpha \iff A^{\mathsf{c}} \in \mathbf{\Sigma}^0_\alpha$$

- $\mathbf{\Sigma}^0_\alpha$ contains all the countable unions of sets in $\mathbf{\Pi}^0_\beta$ for all $\beta < \alpha$, i.e.

$$A \in \mathbf{\Sigma}^0_\alpha \iff A = \bigcup_{n\in\mathbb{N}} A_i \quad \text{with} \quad A_i \in \mathbf{\Pi}^0_{\alpha_i} \quad \text{for some } \alpha_i < \alpha$$

- $\mathbf{\Delta}^0_\alpha = \mathbf{\Pi}^0_\alpha \cap \mathbf{\Sigma}^0_\alpha$

With the point of view of this Hierarchy, we write $\mathbf{\Sigma}^0_1$ for the open sets, the closed sets would be $\mathbf{\Pi}^0_1$ and $\mathbf{\Sigma}^0_2$ is the class of all the countable unions of closed sets.

If X is a polish space, this hierarchy has some interesting properties. For instance, we can show that for each countable ordinal α, $\mathbf{\Sigma}^0_\alpha$ contains a set that is not present in any of the previous class, so the hierarchy "keeps going up" (see Theorem 2.9). It can also be shown that the hierarchy is a well founded order with anti chains of length at most two. In fact, the hierarchy looks like Figure 2.

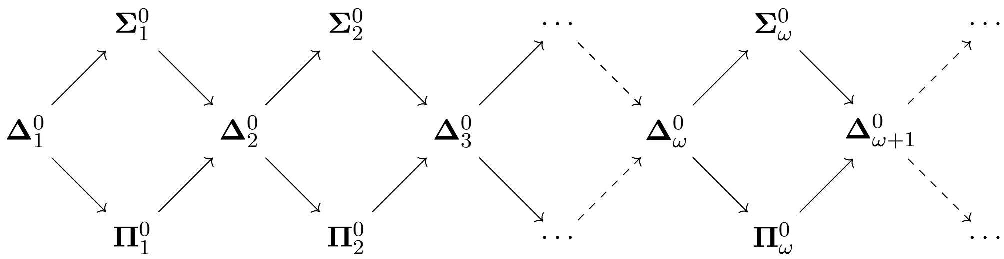


Figure 2: The well-quasi-order of the Baire hierarchy for polish spaces

In the rest of this document we will mostly consider the Baire space and sometimes the Cantor space which are both polish spaces, therefore Figure 2 is an accurate image to have when thinking about the Borel hierarchy.

To understand better the hierarchy, we can look at examples at the lower levels of this hierarchy for the Baire space.

2.1.1 Examples of Borel sets

The $\mathbf{\Pi}^0_1$ sets. The $\mathbf{\Pi}^0_1$ sets are the closed sets. The closed sets in the Baire space and the Cantor set can be described simply as the infinite branches of a tree. To understand why, let $T \subset \omega^{<\omega}$ (resp. $\subset 2^{<\omega}$) be a tree, then $[T] \subset \omega^\omega$ (resp. $\subset 2^\omega$) is a closed set, as all the terminal nodes are associated with basic open sets that are removed from the set of infinite branches[2]:

$$[T] = \omega^\omega \setminus \left(\bigcup_{s \in \omega^{<\omega} \setminus T} [s] \right).$$

Interesting examples of closed sets include:

- the set of sequences that do not contain some integer.
- the set of sequences consisting only of even numbers.
- the set of increasing sequences.

We will soon see a simple way to tell whether or not a set is closed.

The $\mathbf{\Sigma}^0_1$ sets. The $\mathbf{\Sigma}^0_1$ sets are the open sets. A good way of thinking about an open set is to have a correspondence between a set $S \subset \omega^{<\omega}$ and the open set $U = \bigcup_{s \in S}[s]$ generated by S. For instance, if we take $S = \{(1, 2), (1, 3, 4), (17)\}$, it produces the open set in Figure 3.

An important feature of open and closed sets in the context of games is that, if we have a sequence $x \in \omega^\omega$ and we want to know whether or not it belongs to an open set U (or a closet set) by looking only at finite prefixes, then

- if $x \in U$, we can tell with a finite prefix as there must be an integer n such that $[x\restriction_n] \subset U$.
- however if $x \notin U$, we may not be able to know it in finite time. Take for instance $x = (0, 0, \dots)$ and $U = \omega^\omega \setminus \{x\}$. We can't tell that $x \in U$ just by looking at finite prefixes, since at any time, the next digit might very well be non-zero.
- Conversely we can tell in finite time if a sequence does *not* belong to a closed set, but we need infinitely many informations (all its digits) to say that it does belong to a closed set.

[2] Keep in mind that $[\cdot]$ is defined differently for a tree (the infinite branches) and a sequence (the infinite sequences that extend it).

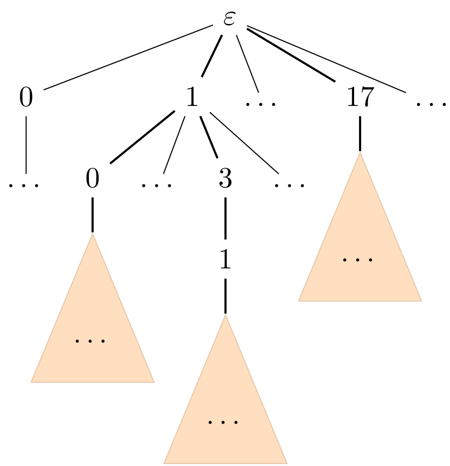


Figure 3: An open set as all the possible extensions of a set of finite sequences. The open sets correspond to all the sequences that are in the orange subtrees.

In practice, this feature allows us to determine if a set X is open or closed by checking if for any x, the membership of x to X of X^{c} can be established with only a finite prefix. For instance, if X is the set of increasing sequences, we cannot tell is a sequence x belongs to X with a finite prefix as any $x{\restriction_n}$ could be continued with a non increasing sequence. However,if $x \notin X$ we can tell it be looking at a finite prefix, since it means that, for some $n \in \omega$, $x_n > x_{n+1}$.

Similarly, the sets of sequences that do not contain a zero is closed: it suffices to see a prefix with a zero to know a sequence doesn't belong to the set.

The $\mathbf{\Delta}^0_1$ sets. The sets at the lowest level of the hierarchy are the clopen sets, which in the Baire space contain all the basic open sets $[t]$, but not only, as there are countably many of those, but $\mathbf{\Delta}^0_1$ has cardinality $2^{\aleph_0}$. Indeed since the basis of any polish space has cardinality $\aleph_0$, they have at most $2^{\aleph_0}$ open sets and the same bound applies for the clopen sets. Thus we only need to construct $2^{\aleph_0}$ clopen sets. In the Baire space, this is a simple task: for each $P \subset \mathbb{N}$, let

$$D_P = \{x \in \omega^\omega \mid x_0 \in P\} = \bigcup_{n \in P} [n]$$

Each of those sets are open since they are the union of open sets, but they are also closed, since their complement $D^{\mathsf{c}}_P = D_{\mathbb{N}\setminus P}$ is open too.

For the Cantor set, the clopen sets have a very nice form. Every open set can be written as the union of basic open sets, but since the Cantor set is compact, any set that is both open and closed can be written as the union of finitely many basic open sets. This means that for any clopen set U there is an integer n and a set $S \subset \{0,1\}^n$ of n-sequences such that

$$U = \{x \in 2^\omega \mid s{\restriction_n} \in S\} = S^\frown 2^\omega$$

That is, the first n digits can be any of a few combinations of zeros and ones but all the following digits can be anything. This also mean that there is only countably many clopen sets in the Cantor set.

Other examples include the sets like

$$X_{n,m} = \{x \in \omega^\omega \mid x_n = m\}$$

of all the sequences that have a m at index n. A simple way to see that the $X_{n,m}$ are clopen sets is to notice that given a sequence $x \in \omega^\omega$, one can always determine with finite information about x whether or not it belongs to $X_{n,m}$ or not, as we can know in finite time if x is in an open set and we can also know in finite time if x is not in a closed set. Since here we only need the n-th digit, $X_{n,m}$ is surely clopen.

The $\mathbf{\Sigma}^0_2$ sets. The $\mathbf{\Sigma}^0_2$ sets are countable unions of closed sets. Examples include $\mathbb{Q} \subset \mathbb{R}$ which is a countable union of (closed) points, but not a countable intersection of open sets, as can be shown by a clever use of the Baire Category theorem. Thus $\mathbb{Q}$ appears first in the hierarchy as a $\mathbf{\Sigma}^0_2$.

An other example in a form that we will see often see is the set of sequence that consist only of zeros after some index:

$$\begin{aligned} X &= \{x \in \omega^\omega \mid \exists n \in \omega,\ \forall m > n,\ x_m = 0\} \\ &= \bigcup_{n \in \omega} \bigcap_{m > n} X_{m,0} \end{aligned}$$

Where $X_{m,0} \in \mathbf{\Delta}^0_1$ is the set of sequences with a zero at the m-th position.

The $\mathbf{\Pi}^0_2$ sets. The $\mathbf{\Pi}^0_2$ sets are countable intersections of open sets, for instance the irrationals in $\mathbb{R}$, isomorphic to the Baire space. Usually, they correspond to quantifying universally over some open set, as for instance the set of sequences with infinitely many zeros,

$$X = \{x \in \omega^\omega \mid \forall n\ \exists m > n x_m = 0\}.$$

We have seen in Proposition 1.9 that the $\mathbf{\Pi}^0_2$ are precisely the subsets of a polish space that can be endowed with a complete metric, and are therefore the only subspace that are polish.

The $\mathbf{\Delta}^0_2$ sets. The $\mathbf{\Delta}^0_2$ are both countable unions of closed sets and countable intersection of open sets. Some examples include the set X of sequences that contain exactly one 0, as it can be described as the intersection of the set of sequences that contain at least one zero, U, which is open and the set of sequences that contain at most one zero, C, which is closed. Since U is open it is also a countable unions of closed sets and thus X is the intersection between a $\mathbf{\Pi}^0_2$ and a closed set, thus $\mathbf{\Pi}^0_2$. On the other hand, since a closed set is also $\mathbf{\Sigma}^0_2$, intersecting with one more open set make X a $\mathbf{\Sigma}^0_2$. Thus X is $\mathbf{\Delta}^0_2$.

An other example of $\mathbf{\Delta}^0_2$ could be sets that contain an even number zeros, but less than 10. One could also argue that this set is more complex than the previous one, and indeed we will discuss this fact in section 5.

The $\mathbf{\Pi}^0_3$ sets. A normal number in the Cantor set, is a number such that any finite sequence consisting of n zeros and ones appears in the number with a proportion of $\frac{1}{2^n}$. More precisely, if $\#_s : 2^{<\omega} \to \omega$ counts the number of times the finite sequence s appears in its input, a number is said to be normal if for all sequences $s \in 2^{<\omega}$, $\lim_{n\to\infty} \frac{\#_s(x\restriction_n)}{n} = \frac{1}{2^{\mathrm{lh}(s)}}$. This captures the idea that all sequences of digits are evenly distributed. The same definition can be made for the reals with their expansion in any base and other sets which are ω-sequence on some finite alphabet.

The set of normal numbers is $\mathbf{\Pi}^0_3$:

$$\mathfrak{N} = \left\{ x \in 2^\omega \,\middle|\, \forall s \in 2^{<\omega}\ \forall \varepsilon \in \omega\ \exists M \in \omega\ \forall m > M\ \left| \frac{S_s(x\restriction_m)}{m} - \frac{1}{2^{\mathrm{lh}(s)}} \right| < \frac{1}{\varepsilon} \right\}$$

with $S_s(x\restriction_m)$ counting the number of occurrences of s in $x\restriction_m$. We will see in Proposition 2.3 how to see in a glimpse that $\mathfrak{N}$ is indeed $\mathbf{\Pi}^0_3$. Other examples of $\mathbf{\Pi}^0_3$ sets are of continuously differentiable functions $C^1([0,1]) \subset C([0,1])$, or the set real sequences that converges to 0.

2.1.2 Some properties

Proposition 2.2. For all countable ordinals α, the Borel classes have the following closure properties:

- All the classes, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$ are closed under finite unions, finite intersections and continuous preimages.
- The $\mathbf{\Sigma}^0_\alpha$ are closed under countable unions.
- The $\mathbf{\Pi}^0_\alpha$ are closed under countable intersections.
- The $\mathbf{\Delta}^0_\alpha$ are closed under complement.

There is a simple way to find to which class belongs a given set provided the set is written in a "canonical" form.

Proposition 2.3. Let $n \in \omega$ be an integer, $S_1, \ldots, S_n$ be countable sets, $v_1, \ldots, v_n$ be variables and $Q_1, \ldots, Q_n \in \{\exists, \forall\}$ be be an alternating sequence of quantifiers. If $P(x, v_1, \ldots, v_n)$ is a "clopen" formula such that for any $v_i \in S_i$ the set $P(x, v_1, \ldots, v_n) \in \mathbf{\Delta}^0_1$ is clopen, then the set

$$X = \{x \in \omega^\omega \,|\, Q_1 v_1 \in S_1\ \ldots\ Q_n v_n \in S_n\ P(x, v_1, \ldots, v_n)\}$$

is

- $\mathbf{\Sigma}^0_n$ if $Q_1 = \exists$.

- $\mathbf{\Pi}^0_n$ if $Q_1 = \forall$.

On can see that P is a clopen formula if determining whether P holds or not can be done by using only a finite prefix of x. Typically, a formula such as $x(n_1) = 0$ is clopen as only the first n_1 digits need to be knowns to know whether or not it holds.

Note that the quantifiers need not to be alternating to use this proposition, as formulas such as $\exists a \in A\ \exists b \in B$ can be written as $\exists (a,b) \in A \times B$ and $A \times B$ is still countable. In this case, the set X is either $\mathbf{\Sigma}^0_n$ or $\mathbf{\Pi}^0_n$ with n the number of alternations between the two quantifiers.

Proof. The proof is done by induction on the number of quantifiers, as one can notice that

$$X = \{x \in \omega^\omega \mid \exists n \in S\ P(x,n)\} = \bigcup_{n \in S} \{x \in \omega^\omega \mid P(x,n)\},$$

and similarly,

$$X = \{x \in \omega^\omega \mid \forall n \in S\ P(x,n)\} = \bigcap_{n \in S} \{x \in \omega^\omega \mid P(x,n)\},$$

therefore, when quantifying existentially over some $\mathbf{\Pi}^0_\alpha$, one obtains a $\mathbf{\Pi}^0_{\alpha+1}$, and quantifying universally over a $\mathbf{\Pi}^0_\alpha$ yields a $\mathbf{\Sigma}^0_{\alpha+1}$. □

Proposition 2.4. Let α be a countable ordinal. Then the class $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$ have cardinality $2^{\aleph_0}$.

Proof. We have seen in the examples that the class $\mathbf{\Delta}^0_1$, $\mathbf{\Sigma}^0_1$ and $\mathbf{\Pi}^0_1$ have the cardinality of the reals, $2^{\aleph}_0$. Furthermore, each class is corresponds to all the possible countable unions or intersections of sets in the lower classes. It suffices to show the proposition for the $\mathbf{\Sigma}^0_\alpha$ classes as the $\mathbf{\Pi}^0_\alpha$ class is its dual and $\mathbf{\Delta}^0_\alpha$ is contained in $\mathbf{\Sigma}^0_\alpha$ and contains $\mathbf{\Sigma}^0_\beta$ for $\beta < \alpha$. Let $\mathbf{\Gamma} = \bigcup_{\xi<\alpha} \mathbf{\Pi}^0_\xi$. Since α is countable and by our induction hypothesis the classes $\mathbf{\Pi}^0_\xi$ have cardinality $2^{\aleph_0}$, $\mathbf{\Gamma}$ also has cardinality $2^{\aleph_0}$. Now $\mathbf{\Sigma}^0_\alpha$ has cardinality at most

$$|{}^\omega\mathbf{\Gamma}| = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}.$$

□

Corollary 2.5. The Borel hierarchy $\mathcal{B} = \bigcup_{\alpha \in \omega_1} \mathbf{\Sigma}^0_\alpha$ contains $2^{\aleph_0}$ sets.

Proof. Since each class $\mathbf{\Sigma}^0_\alpha$ contains $2^{\aleph_0}$ sets and there are ω_1 of them, we have $|\mathcal{B}| = 2^{\aleph_0} \cdot \aleph_1$ which is $2^{\aleph_0}$ since $2^{\aleph_0} \leq \omega$. □

We will need the following two proportion which are useful representations of Borel sets and Polish spaces.

Proposition 2.6. Let $A \subset \omega^\omega$ be a non-empty Borel set, then there is a continuous surjection $f : \omega^\omega \to A$.

Proof. The proof of a more general result is Theorem 13.7 in [Kechris, 2012]. □

Proposition 2.7. Let X be a non-empty Polish space, then there is a continuous surjection $f : \omega^\omega \to X$.

Proof. The proof of a more general result is Theorem 13.7 in [Kechris, 2012]. □

Most examples of sets that come from analysis are most of the time either of rank lower than 5, or not Borel. Therefore to understand what the levels are and that the inclusions are proper, we need to construct new sets.

2.1.3 Universal sets

Definition 2.8. Let X be a polish space and $\mathbf{\Gamma}$ any class of sets for instance a $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, Borel or analytic. We write $\mathbf{\Gamma}(X)$ for the sets that are $\mathbf{\Gamma}$ in X.

A set $U \subset X \times Y$ is called Y**-universal** if U is in the class $\mathbf{\Gamma}(X \times Y)$ and for all $y \in Y$, the sets $U_y = \{x \in X \mid (x, y) \in U\}$ enumerate all sets of $\mathbf{\Gamma}(X)$.

The U_y provide a parametrisation for all the $\mathbf{\Gamma}$ sets on X.

Note that this definition is particularly interesting when the space $X \times Y$ is homeomorphic to X, which is the case for instance for the cantor set and the Baire space when one considers

$$2^\omega \times 2^\omega \simeq 2^\omega \quad \text{and} \quad \omega^\omega \times \omega^\omega \simeq \omega^\omega,$$

and even that

$$(2^\omega)^{\mathbb{N}} \simeq 2^\omega \quad \text{and} \quad (\omega^\omega)^{\mathbb{N}} \simeq \omega^\omega.$$

Indeed, in this case universal sets do not need to live outside of the space X, be we can have some X-universal sets as subsets of X, provided an homeomorphism between X^2 and X. We will now constructs an explicit set that belongs precisely to each $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$. In fact it will be even better as we constructs universal sets that belong to each class and not before.

Theorem 2.9. Let X be a Polish space. For all ordinal $\alpha < \omega_1$, there exists a 2^ω-universal set for $\mathbf{\Sigma}^0_\alpha$ and its complement is 2^ω-universal for $\mathbf{\Pi}^0_\alpha$.

Furthermore, if X is uncountable, $\mathbf{\Sigma}^0_\alpha \neq \mathbf{\Pi}^0_\alpha$.

Proof. We proceed by induction on the ordinals, so we start by constructing a universal open set. To that end, fix any enumeration $(V_i)_{i\in\omega}$ of the basic open sets of X. All open sets (countable) unions of them, so we can encode this with a sequence $y \in 2^\omega$ this way:

$$U_y = \bigcup_{\substack{n\in\omega \\ y(n)=1}} V_n.$$

Our universal open set is then

$$\mathcal{U} = \{(x, y) \in X \times 2^\omega \mid x \in U_y\},$$

or equivalently $\mathcal{U} = \{x \in X \times 2^\omega \mid \exists n \in \mathbb{N}\ y(n) = 1 \wedge x \in V_n\}$ which is clearly open as a countable union of open sets. Notice that the complement $\mathcal{F}$ of $\mathcal{U}$ is a universal closed set as the complements of the U_is enumerate all the closed sets of X.

For the recursive case, suppose we have $\mathcal{F}_\xi$ a universal $\mathbf{\Pi}^0_\xi$ for all $\xi < \alpha$. We want to construct a universal $\mathbf{\Sigma}^0_\alpha$ that should enumerate all $\mathbf{\Sigma}^0_\alpha$ which are countable unions of $\mathbf{\Pi}^0_\xi$ for $\xi < \alpha$. Since every set in $\mathbf{\Pi}^0_\xi$ is represented by an element of the Cantor set, we can represent an element of $\mathbf{\Sigma}^0_\alpha$ by a sequence of elements in the Cantor set. To that extent, fix an increasing sequence $\xi_n < \alpha$ of ordinals[3] such that $\bigcup_{n\in\omega}(\xi_n + 1) = \alpha$. We will consider only unions with elements taken in the $\mathbf{\Pi}^0_{\xi_n}$, as all unions can be written in this form. Our universal set is then

$$\mathcal{U}_\alpha = \left\{(x, (y_n)_{n\in\omega}) \in X \times (2^\omega)^\omega \,\middle|\, \exists m \in \omega\ (x, y_m) \in \mathcal{F}_{\xi_n}\right\}$$

Note that $\mathcal{U}_\alpha$ isn't a subset of $X \times 2^\omega$, but since $X \times (2^\omega)^\omega$ is homeomorphic to $X \times 2^\omega$, we can take its image by any homeomorphism.

To show that the $\mathbf{\Sigma}^0_\alpha \neq \mathbf{\Pi}^0_\alpha$, we use a diagonal argument. Since X is uncountable, we can assume that $2^\omega \subset X$. We now consider the set

$$A = \{y \in 2^\omega \mid (y, y) \notin \mathcal{U}_\alpha\}.$$

We have that $A \in \mathbf{\Pi}^0_\alpha$, since it is the preimage of $\mathcal{U}^c_\alpha = \mathcal{F}_\alpha$ through the continuous function $y \mapsto (y, y)$. However, if $\mathbf{\Pi}^0_\alpha = \mathbf{\Sigma}^0_\alpha$, we would have $A \in \mathbf{\Sigma}^0_\alpha$ and thus a code $y_0 \in 2^\omega$ such that $A = \{x \in 2^\omega \mid (x, y_0) \in \mathcal{U}_\alpha\}$, which is a contradiction with the definition of A. Thus $A \notin \mathbf{\Pi}^0_\alpha$ and the ω_1 levels of the Borel hierarchy are proper subsets of each other. □

2.2 Wadge Hierarchy

An other, much finer approach to the classification of sets is the Wadge hierarchy. This hierarchy is based on an idea very often seen in computer science of reducing a problem to an other problem via a specified kind of function. Here, we will reduce a set to an other via preimages of continuous functions. More precisely we say that A reduces to B if there exists a continuous function f such that $f^{-1}(B) = A$. If it is the case, we write $A \leq_{\mathrm{W}} B$ and read "A reduces to B".

This corresponds to the idea of complexity in the following sense: if we consider that continuous functions are simple to evaluate, then determining membership in A is not harder than determining membership in B, as if we want to know whether $x \in A$, we can just compute $f(x)$ and check if it belongs to B.

$$x \in A \iff f(x) \in B$$

[3]Note that we use here more than the axiom of countable choice, as we choose ω_1 cofinal sequences.

Furthermore, the supposition that continuous functions are "simple" to evaluate isn't surprising as we often see them as the simplest functions (after maybe, constant or Lipschitz functions), but more interestingly they precisely correspond to functions that can be calculated by a Turing machine equipped with an oracle $x \in \omega^\omega$ containing information about the preimages of basic open sets. This matter is discussed in great detail in [Wadge, 1982, pp. 27–32].

Definition 2.10. Let $A, B \subset X$ a polish space. We say that A **reduces** to B if there is a continuous function $f : X \to X$ such that $f^{-1}(B) = A$.

We write $A \leq_{\mathrm{W}} B$ if A reduces to B and otherwise $A \not\leq_{\mathrm{W}} B$. Similarly, we write $A \geq_{\mathrm{W}} B$, if B reduces to A and $B \not\geq_{\mathrm{W}} A$ otherwise.

If both $A \leq_{\mathrm{W}} B$ and $B \leq_{\mathrm{W}} A$, we say that A and B are **Wadge equivalent** and write $A \equiv_{\mathrm{W}} B$.

The relation $\equiv_{\mathrm{W}}$ is an equivalence relation that induces **Wadge degrees**. We write

$$[A]_{\equiv_{\mathrm{W}}} = \{B \subset X \mid A \equiv_{\mathrm{W}} B\}$$

Unlike the Borel hierarchy, the shape of this hierarchy is not clear at first sight, but we will show that they are very similar. In fact, the Wadge hierarchy on the Borel sets has the same structure as depicted in Figure 2 and coincides nicely with the Borel hierarchy. However, the Wadge hierarchy is much, much finer. For instance, the $\mathbf{\Sigma}^0_2$ sets correspond to the ω_1 first degrees of the Wadge hierarchy. Afterwards, the $\mathbf{\Sigma}^0_3$ correspond the $\omega_1^{\omega_1}$ first Wadge degrees, and likewise, for $n \in \omega$, $\mathbf{\Sigma}^0_n$ correspond to the

$$\underbrace{\omega_1^{\omega_1^{\cdot^{\cdot^{\omega_1}}}}}_{n \text{ times}}$$

first degrees.

More generally, we proved that the Borel hierarchy goes for ω_1 degrees but on the Borel sets, the Wadge hierarchy has more degrees: let $\varphi_0(\alpha) := \omega_1^\alpha$ be the ordinal exponentiation of base ω_1. We construct φ_ξ by induction on the ordinals as the function that enumerate the fixed points of cofinality ω_1 of all the previous functions. For instance,

$$\alpha = \omega_1^{\omega_1^{\omega_1^{\cdot^{\cdot^{\cdot}}}}}$$

is the first fixed point of φ_0, but it is of cofinality ω. It turns out this ordinal is the size of the Wadge hierarchy on the union of the $\mathbf{\Sigma}^0_n$, for $n \in \omega$. However, the size of the Wadge hierarchy on the whole Borel sets is $\varphi_{\omega_1}(0)$, the first fixed point of the all the first ω_1 functions.

2.2.1 Duality

We set a few definitions that will prove useful to understand the different classes and their relations.

Definition 2.11. Let $\mathbf{\Sigma}^0_b$ e any class of sets. The **dual class** of $\mathbf{\Gamma}$ is the class of complements of sets in $\mathbf{\Gamma}$ and is noted by $\check{\mathbf{\Gamma}}$ (read "gamma tchetch"). We have

$$\check{\mathbf{\Gamma}} = \{X^{\mathsf{c}} \mid X \in \mathbf{\Gamma}\}$$

For instance, the class $\mathbf{\Sigma}^0_\alpha$ is the dual of $\mathbf{\Pi}^0_\alpha$, by definition. An other pair of dual classes, in the Wadge hierarchy this time are the classes $[\varnothing]_{\equiv_{\mathrm{W}}}$ and $[\omega^\omega]_{\equiv_{\mathrm{W}}}$.

However some classes are their own dual class, for instance all the $\mathbf{\Delta}^0_\alpha$, as if X is both $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$, its complement belongs also to $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Sigma}^0_\alpha$, and thus is $\mathbf{\Delta}^0_\alpha$. This motivates the following distinction:

Definition 2.12. A class of sets $\mathbf{\Gamma}$ is called **self-dual** if $\check{\mathbf{\Gamma}} = \mathbf{\Gamma}$. If it is not the case, we call $\mathbf{\Gamma}$ a **non-self-dual** class.

We extend this definition to sets in the Wadge hierarchy and not only to classes.

Definition 2.13. A set $X \subset \omega^\omega$ is called **self-dual** if it is in same Wadge degree than its complement, that is, if $[X^{\mathsf{c}}]_{\equiv_{\mathrm{W}}} = [X^{\mathsf{c}}]_{\equiv_{\mathrm{W}}}$. On the other hand, if $[X^{\mathsf{c}}]_{\equiv_{\mathrm{W}}} \neq [X]_{\equiv_{\mathrm{W}}}$, we call X **non-self-dual**.

We have seen that $\varnothing$ is non-self-dual as it does not reduce to its complement ω^ω. However the clopen set of sequences that start with a one is self-dual, as it reduces to its complement through the continuous function

$$\begin{array}{rccl} f: & \omega^\omega & \longrightarrow & \omega^\omega \\ & x & \longmapsto & \begin{cases} (0,0,\dots) & \text{if } x_0 = 1 \\ (1,1,\dots) & \text{if } x_0 \neq 1 \end{cases} \end{array}$$

2.2.2 Complete sets

Definition 2.14. Let X be a polish space, and $\mathbf{\Gamma}$ a class of subsets of X. We say that $\mathbf{\Gamma}$ is an **initial class** if for all $A, B \subset X$ with $A \leq_{\mathrm{W}} B$ and $B \in \mathbf{\Gamma}$ then we also have $A \in \mathbf{\Gamma}$.

Examples of initial classes include all the $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$, but also the class of Borel sets, of analytic sets... Indeed all those classes are closed under continuous preimages.

Proposition 2.15. The only Wadge degrees that are initial classes are the degrees at the bottom of the Wadge hierarchy, that is, $[\varnothing]_{\equiv_{\mathrm{W}}}$ and $[X]_{\equiv_{\mathrm{W}}}$.

Proof. First we notice that $[\varnothing]_{\equiv_\mathrm{w}} = \{\varnothing\}$ as the preimage of the empty set by any function is the empty set. This also show that it is an initial class as it is trivially closed under preimages. Similarly, $[X]_{\equiv_\mathrm{w}} = \{X\}$ and $[X]_{\equiv_\mathrm{w}}$ is an initial class.

Note that those two degree are thus incomparable, that is $\varnothing \not\leq_\mathrm{w} X$ and $X \not\leq_\mathrm{w} \varnothing$. We have incidentally shown that the Wadge hierarchy is not a well order.

We now show that they are the only initial degrees, that is, for any set $A \subset X$ with $A \neq \varnothing$ and $A \neq X$, we have both $\varnothing \leq_\mathrm{w} A$ and $X \leq_\mathrm{w} A$. It suffices to show that the constant map $f : x \mapsto a$ is a reduction to $\varnothing$ if $a \notin A$ and a reduction to X if $a \in A$. □

Definition 2.16. Let X be a polish space, $C \subset X$ be a set, and $\mathbf{\Gamma}$ be any class or subset of X. We say that C is **$\mathbf{\Gamma}$-hard** if for all $A \in \mathbf{\Gamma}$, we have $A \leq_\mathrm{w} C$. Moreover if $C \in \mathbf{\Gamma}$, we say that C is **$\mathbf{\Gamma}$-complete**.

With this terminology taken directly from computer science, a set is $\mathbf{\Gamma}$-hard if it reduces to all the sets of $\mathbf{\Gamma}$, therefore determining membership in a $\mathbf{\Gamma}$ set is at most as hard as determining membership in C. If C is $\mathbf{\Gamma}$-complete, it is a good representant of the complexity of $\mathbf{\Gamma}$ as sets in $\mathbf{\Gamma}$ are at most as complex as C.

Not every class has a complete set, and we will see that it is not the case for $\mathbf{\Delta}^0_2$ for instance.

Proposition 2.17. Let X be a polish space and $C \subset X$ be a $\mathbf{\Gamma}$-complete set, for a given class $\mathbf{\Gamma}$. Then its complement C^c is $\check{\mathbf{\Gamma}}$-complete.

Proof. Let $A \in \check{\mathbf{\Gamma}}$ be a set. Since $A^\mathsf{c} \in \mathbf{\Gamma}$, we have that $A \leq_\mathrm{w} C$, and therefore there is a function $f : X \to X$ such that $f^{-1}(C) = A^\mathsf{c}$. But then $f^{-1}(C^\mathsf{c}) = f^{-1}(C)^\mathsf{c} = A$. □

Proposition (3.8)**.** The set of sequences that contain an infinite amount of zeros,

$$X = \{x \in \omega^\omega \mid \exists^\infty n \; x(n) = 0\}$$

is $\mathbf{\Pi}^0_2$ complete. Its complement, the set of sequences that contain finitely many zeros is $\mathbf{\Sigma}^0_2$-complete.

We will however delay the proof of this fact, as proving results about the Wadge hierarchy will be much simpler once we have developped infinite games.

2.3 Borel's hierarchy for functions

The inspiration for this hierarchy of functions is the definition of continuous functions, for which the preimage of an open set must be an open set, that is if U is open, $f^{-1}(U)$ is open too. There are two different ways of extending this notion. We can either loosen it and allow for $f^{-1}(U)$ to be a more complex set, for instance any class that we defined previously, or we can strengthen it it by considering other classes for U. In fact, we will do both, as the second way is not interesting without the first one.

We start by extending the possibilities for the preimages.

Definition 2.18 (Baire class)**.** We define the notion of **Baire class** α by induction on the countable ordinals.

For the base case, a function is Baire class 0 if it is a continuous function. For α an ordinal, a function $f : X \to X$ is Baire class α if it is the pointwise limit of functions of Baire class less than α, that is, there exist $(f_n)_{n\in\omega} : X \to X$ and $(\alpha_n)_{n\in\omega} < \alpha$ such that, for all $x \in X$,

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{and} \quad f_n \text{ is Baire class } \alpha_n < \alpha.$$

Notice that the hierarchy stops growing at ω_1, and no new function is added. Indeed, if we take a function $f = \lim f_n$ where each f_n is of Baire class $\alpha_n < \omega_1$, then f is Baire class α, where $\alpha = \bigcup a_n$. However, $\alpha < \omega_1$ since each α_n is countable and ω_1 is not a countable union of countable ordinals.

Proposition 2.19. A function $f : \omega^\omega \to \omega^\omega$ is Baire class α if and only if for every open set $U \subset X$, $f^{-1}(U) \in \mathbf{\Sigma}^0_{\alpha+1}$.

Proof. We prove this result by induction on α. The base case follows from the definition, and for recursive case α, let f be a function of Baire class α and f_n of Baire class $\alpha_n < \alpha$ such that $f = \lim f_n$ is their pointwise limit. By our induction hypothesis, for all n, $f_n^{-1}(U) \in \mathbf{\Sigma}^0_{\alpha_n}$. At the same time, we notice that the condition on the preimage can be stated equivalently for closed sets, since

$$f_n^{-1}(U) \in \mathbf{\Sigma}^0_{\alpha_n} \iff f_n^{-1}(U^\mathsf{c}) = f_n^{-1}(U)^\mathsf{c} \in \mathbf{\Pi}^0_{\alpha_n}.$$

This means that if we take a basic open set $[t]$, which is both open and closed, we have $f_n^{-1}([t]) \in \mathbf{\Sigma}^0_{\alpha_n} \cap \mathbf{\Pi}^0_{\alpha_n} = \mathbf{\Delta}^0_{\alpha_n}$. If we now look at which class is $f^{-1}([t])$, we obtain

$$\begin{aligned} f^{-1}([t]) &= \{x \in X | \exists n \in \omega,\ \forall m > n,\ f_m(x) \in [t]\} \\ &= \bigcup_{n\in\omega} \bigcap_{m>n} \{x \in X \mid f_m(x) \in [t]\} \\ &= \underbrace{\bigcup_{n\in\omega} \underbrace{\bigcap_{m>n} \underbrace{f_m^{-1}([t])}_{\in \mathbf{\Delta}^0_\alpha}}_{\in \mathbf{\Pi}^0_\alpha}}_{\in \mathbf{\Sigma}^0_{\alpha+1}} \end{aligned}$$

So $f^{-1}([t])$ is a $\mathbf{\Sigma}^0_{\alpha+1}$, and since any open set U is a countable union of basic open sets, their preimage $f^{-1}(U)$ is a countable union of $\mathbf{\Sigma}^0_{\alpha+1}$, which is still a $\mathbf{\Sigma}^0_{\alpha+1}$.

The proof of the other direction is a special case of Theorem 24.3 in [Kechris, 2012]. □

This shows that the Baire classes are indeed the first generalisation mentioned at the beginning of the section. Having the two viewpoints on those classes yields an immediate corollary.

Definition 2.20. A function $f : X \to X$ is a **Borel function** if the preimage of every open set through f is a Borel set.

Corollary 2.21. The set of all Borel functions on ω^ω is the smallest set that contains the continuous functions and that is closed under pointwise limits.

Equivalently, a function is Borel if and only if it Baire class α, for some ordinal α.

Proof. All the Baire classes are clearly contained in the set of Borel functions, so only have to show the other direction. Let $f : \omega^\omega \to \omega^\omega$ be a Borel function. The fact that the preimage by f of any open set if a Borel set is equivalent to the fact that the preimage of any basic open set is Borel. However, there are only countably many basic open sets, so for an open set $[t]$ we write α_t the least ordinal such that $f^{-1}([t]) \in \mathbf{\Sigma}^0_{\alpha_t}$. Therefore the preimage of any basic set is in $\mathbf{\Sigma}^0_\alpha$ with $\alpha = \bigcup_{t \in \omega^{<\omega}} \alpha_t$, but since this is a countable union, $\alpha < \omega_1$ and therefore f is Baire class α. □

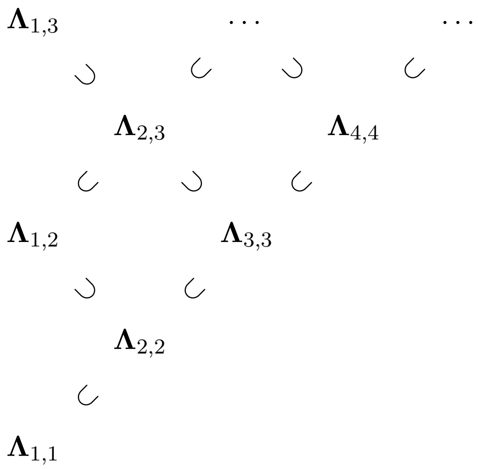
The other way to generalise the notion of continuous functions function is to extend both the class from which we consider preimages and the class to which we restrict them.

Definition 2.22. We denote by $\mathbf{\Lambda}_{n,m}$ the class of functions $f : \omega^\omega \to \omega^\omega$ such that for any $A \in \mathbf{\Sigma}^0_m$, $f^{-1}(A) \in \mathbf{\Sigma}^0_n$.

$$\mathbf{\Lambda}_{n,m} = \left\{ f : \omega^\omega \to \omega^\omega \,\middle|\, \forall A \in \mathbf{\Sigma}^0_m \; f^{-1}(A) \in \mathbf{\Sigma}^0_n \right\}$$

With this notation, the continuous functions correspond to $\mathbf{\Lambda}_{1,1}$ as the class of open sets is $\mathbf{\Sigma}^0_1$. By Proposition 2.19, the classes $\mathbf{\Lambda}_{1,n}$ correspond precisely to the functions Baire class $n+1$. The functions $\mathbf{\Lambda}_{2,2}$ are the function such that the preimage of a countable union of closed set is a countable union of closed sets, therefore it must contain the $\mathbf{\Lambda}_{1,1}$ since if C_n is a collection of closed sets and $f \in \mathbf{\Lambda}_{1,1}$, we have $f^{-1}(\bigcup C_n) = \bigcup f^{-1}(C_n)$ which is a countable union of closed sets.

More generally we have the following picture:



Proposition 2.23. Let m and n be two ordinals. Then

- $\mathbf{\Lambda}_{m,n} \subset \mathbf{\Lambda}_{m+1,n+1}$.
- $\mathbf{\Lambda}_{m,n} \subset \mathbf{\Lambda}_{m-1,n}$ if m is successor.

Proof. Let $f \in \mathbf{\Lambda}_{m,n}$ be a function such that for all $A \in \mathbf{\Sigma}^0_m$, $f^{-1}(A) \in \mathbf{\Sigma}^0_n$. First we can see that the preimage of sets in $\mathbf{\Pi}^0_m$ are in $\mathbf{\Pi}^0_n$ as the preimage and the complement commutes. But since the operation of countable union and preimages also commutes, the preimage of elements in $\mathbf{\Sigma}^0_{m+1}$ is $\mathbf{\Sigma}^0_{n+1}$.

For the other point, we notice that $\mathbf{\Sigma}^0_m \supset \mathbf{\Sigma}^0_{m-1}$, therefore requesting that the preimages of $\mathbf{\Sigma}^0_m$ to be $\mathbf{\Sigma}^0_n$ is broader that requesting the preimages of $\mathbf{\Sigma}^0_{m-1}$. □

We will only study the lower levels of this hierarchy in subsection 4.4.

2.4 Analytic sets

Borel sets are not the most complex sets that we study, as above the Borel hierarchy lies the projective hierarchy, consisting of classes $\mathbf{\Sigma}^1_\alpha$, $\mathbf{\Pi}^1_\alpha$, and $\mathbf{\Delta}^1_\alpha$.

We will not cover this hierarchy in its whole, but instead look only at what lies at its lowest levels, the analytic sets, as we will need one important result — Lusin's theorem — to prove other results about the Borel hierarchy.

Definition 2.24. Let X be a polish space. A set $A \subset X$ is **analytic** if there is a polish space Y and a continuous function $f : Y \to X$ such that $A = f(Y)$. In other words, A is the continuous image of a polish space.

There are a lot of equivalent definitions for analytic sets, and some will be more practical depending on the context.

Proposition 2.25. For a set $A \subset \omega^\omega$, the following conditions are equivalent:

1. A is analytic.
2. A is a continuous image of the Baire space (if $A \neq \varnothing$).
3. A is the projection along a coordinate of a closed set of $\omega^\omega \times \omega^\omega$.
4. A is the projection along a coordinate of a Borel set of $\omega^\omega \times \omega^\omega$.

Proof. We only consider the cases where A is non empty, as the empty set satisfies all the conditions.

- 2. $\implies$ 1. and 3. $\implies$ 4. are clear.
- 1. $\implies$ 2. By Proposition 2.7 there is a surjection from ω^ω to any non empty polish space, and by composition A must be a continuous image of the Baire space.
- 2. $\implies$ 3. Let $f : \omega^\omega \to \omega^\omega$ be a continuous function such that $f(\omega^\omega) = A$, we can consider the graph $G \subset \omega^\omega \to \omega^\omega$ if f which is a closed set as f is continuous. But then the range of f is precisely A and therefore $\pi_2(G) = A$.
- 3. $\implies$ 1. If $A = \pi_1(B)$ with B a closed subset of $\omega^\omega \times \omega^\omega$ and where π_1 is the projection on the first coordinate, then by Proposition 1.9, B is a polish space and $\pi_1\restriction_B$ is a continuous function, thus A is analytic.
- 4. $\implies$ 3. Let $B \subset \omega^\omega \times \omega^\omega$ be a Borel set such that $\pi_1(B) = A$. By Proposition 2.6 we have a surjection $f : \omega^\omega \to B$. Now the function $\pi_1 \circ f : \omega^\omega \to \omega^\omega$ is a continuous function, therefore its graph $G \subset \omega^\omega \times \omega^\omega$ is closed. We claim: $\pi_2(G) = A$, indeed

$$\begin{aligned}\pi_2(G) &= \{y \in \omega^\omega \mid \exists x \in \omega^\omega \ (x, y) \in G\} \\ &= \{y \in \omega^\omega \mid \exists x \in \omega^\omega \ \pi_1(f(x)) = y\} \\ &= \{y \in \omega^\omega \mid \exists z \in B \ \pi_1(z) = y\} && \text{since } f \text{ is surjective} \\ &= \pi_1(B) = A.\end{aligned}$$

□

Definition 2.26. Let A, B and S be three sets. We say that S **separates** A and B, if $A \subset S$ and $B \cap S = \varnothing$

If for a given A and B, there exists a such S that is a Borel set, we say that A and B are **Borel-separable**.

Lemma 2.27. Let $P = \bigcup_n P_n$ and $Q = \bigcup_m Q_m$ be two countable unions of sets, such that P_n and Q_m are Borel-separable for all n and m, then P and Q are Borel-separable.

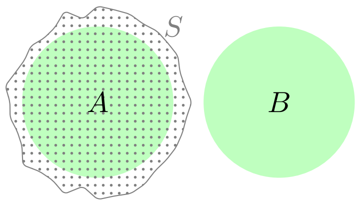


Figure 4: S separates A and B

Proof. Suppose that P_n and Q_m are separated by the Borel set $S_{n,m}$, that is $P_n \subset S_{n,m}$ and $Q_m \cap S_{n,m} = \varnothing$. We set

$$S = \bigcup_{n\in\omega} \bigcap_{m\in\omega} S_{n,m}.$$

We claim that P and Q are separated by S, because

$$S = \bigcup_{n\in\omega} \bigcap_{m\in\omega} S_{n,m} \subset \bigcup_{n\in\omega} \bigcap_{m\in\omega} P_n = \bigcup_{n\in\omega} P_n = P$$

and on the other hand, let $m' \in \omega$. Since $S_{n,m'} \cap Q_{m'} = \varnothing$, we have

$$S \cap Q_{m'} = \bigcup_{n\in\omega} \bigcap_{m\in\omega} S_{n,m} \cap Q_{m'} = \varnothing$$

Thus, $S \cap Q = \varnothing$. Since S is a Borel set, P and Q are Borel-separable. □

Theorem 2.28 (Lusin)**.** Let X be a polish space. If $A, B \subset X$ are two disjoints analytic sets, then they are Borel-separable.

Proof. Without loss of generality, we can assume that A and B are non-empty, so let $f : \omega^\omega \to A$ and $g : \omega^\omega \to B$ be two continuous surjections. For any basic open set $[s]$ of ω^ω, we write $A_s = f([s])$ and $B_s = f([s])$. So that we always have

$$A_s = \bigcup_{n\in\mathbb{N}} A_{s^\frown n} \quad \text{and} \quad B_s = \bigcup_{n\in\mathbb{N}} B_{s^\frown n}.$$

In particular, A and B are countable unions. In order to find a contradiction assume that A and B are not Borel-separable. By the Lemma 2.27, there exists x_0 and y_0 such that A_{x_0} and B_{y_0} are not Borel-separable. Continuing the process, we obtain two sequences $(x_n), (y_n) \in \omega^\omega$ such that, for all n, $A_{x\restriction n}$ and $B_{y\restriction n}$ are not Borel-separable.

Now, $f(x) \in A$ and $g(y) \in B$, which are disjoint, so $f(x) \neq g(x)$. Let $U_x, U_y \subset X$ be two disjoint open sets containing $f(x)$ and $f(y)$ respectively. By continuity of f and g, there exists an integer n large enough such that $f([x\restriction n]) \subset U_x$ and $g([y\restriction n]) \subset U_y$, but then U_x separates $A_{x\restriction n}$ from $B_{y\restriction n}$, which is our contradiction. □

3 Infinite games

3.1 Games and strategies

We will study games between two players, that we call Player I and Player II. Those games will be turn-based and with complete information, like chess, but that can (and will) be infinitely long. We see games as trees, where each vertex is a configuration, for instance, the board at a given turn in chess, and an edge between u and v correspond to a move by one of the two players from the configuration u to the configuration v. This way, the root of the tree is the initial configuration of the game, and each turn corresponds to moving down the tree.

Definition 3.1. Let X be a set. An **infinite game** on X is a pair (T, W), where $T \subset X^{<\omega}$ is a non-empty pruned tree and $W \subset [T]$ is a subset of the infinite branches of T. We call X the set of **moves**, and W the **winning set**.

A **play** of the game is a sequence $s = (s_0, s_1, \dots) \in [T]$ such that

- $s_0 \in \mathrm{succ}(\varepsilon) \subset T$, is the first move of Player I and is any successor of the root of the game tree T.
- On step $2n + 1$, Player II picks the next move, $s_{2n+1} \in \mathrm{succ}(s_{2n})$.
- Then on step $2n$, Player I picks the next move, $s_{2n} \in \mathrm{succ}(s_{2n-1})$

In the end, we say that Player II wins if $s \in W$. Otherwise, Player I wins.

We note that for a play s of a game, $\mathrm{even}(s)$ are the moves made by Player I and $\mathrm{odd}(s)$ are the moves made by Player II.

We often see a game between two players as both players pushing down a penny in the game tree. That is, we imagine that at the start of the game, we put a penny at the root of the tree, and every turn a player chooses towards which successor he wishes to push that penny. A play of the game is then just the sequence of nodes the penny has visited, which may belong to the winning set or not.

Definition 3.2. Let A, B be such that $\{\mathtt{A}, \mathtt{B}\} = \{\text{Player } \mathtt{I}, \text{Player } \mathtt{II}\}$. A **strategy** in the game (T, W) for B is a tree $S \subset T$ such that, for all $s \in S$:

- on A's turn, we have $\mathrm{succ}_S(s) = \mathrm{succ}_T(s)$. Where $\mathrm{succ}_S(s)$ is the set of successors of s in S. This means that A can play any move that is legal.
- on B's turn, we have $\mathrm{succ}_S(s) = \{x\}$, where $x \in \mathrm{succ}_T(s)$ is any successor of s. This correspond to the idea that B already knows what to play in every case.

It is his strategy.

An other way to view strategies is to define it as a **strategy function**, that is, a function τ that associate to each finite sequence of moves of one player the next move of the other player.

Those to notions are equivalent, indeed, given a strategy tree T for Player `II` (say), we can construct a strategy function $\tau : X^{<\omega} \to X$ that maps a finite play of Player `I` to the answer of Player `II`, as

$$\tau(x) = y \iff \exists z \in T \text{ s.t. } \begin{cases} \text{even}(z) = x \\ \wedge \\ \text{last}(z) = y \\ \wedge \\ \text{lh}(z) = 2 \cdot \text{lh}(x) \end{cases}$$

The last condition ensures that both players have played the same number of moves and since Player `I` starts, y is a move made by Player `II`. Note that τ takes only a sequence of moves from Player `I` as the answers from Player `II` at each of the previous turns can be computed with τ. Alternatively, given a strategy function, we can define a strategy tree. We will use either point of view depending on what makes proving easier.

The point of view of strategy functions has the advantage of focusing on the immediate succession of moves, as "Player `I` plays s, then Player `II` plays $\tau(s)$". This will make defining strategies easier since this correspond to the way we play games, for instance in chess, one might say "If you play `e4`, I play `d5`". However it is slightly less efficient to handle whole plays of a game. Indeed, given a strategy tree T, an instance of the game where the player follows his strategy is just a sequence $x \in T$. With a strategy function we need to "build back" the whole play from all the individual moves. In order to make this simpler we will usually define an **extended strategy function** $\tilde{\tau}$ from a strategy τ. This function will take a whole play of (say) Player `I` and return the whole play produced by Player `II`. Most of the time, if x is an infinite play of Player `I` against τ, we will define $\tilde{\tau}(x)$ as the sequence of Player `II`'s moves, that is,

$$\tilde{\tau}(x) = \big(\tau(x\restriction_1), \tau(x\restriction_2), \tau(x\restriction_3), \dots\big)$$

It may however depend on the nature of the game, and this definition may not be the most appropriate. If it is the case, we will make sure to precise the definition.

Strategies in themselves are not very interesting, as they are just subtrees, and we will instead focus on winning strategies, strategies that guaranty one player to win, no matter what the other player does.

Definition 3.3. A **winning strategy** for a given player in the game (T, W) is a strategy $S \subset T$ such that all the infinite branches of S are games that this player wins, that is, $[S] \subset W$.

Having formally introduced games and strategies, we now consider examples of games, most of them that will take place in the Baire space and that we will use as *tools* in the next sections.

3.2 Traditional infinite games

Perhaps the simplest interesting game that can be played on the Baire space, is the Gale-Stewart game.

Definition 3.4. Let $W \subset \omega^\omega$ be any set. The **Gale-Stewart game** of W, $G(W)$ is defined as follows.

At each turn, each player play an integer so that they build together a sequence $x = (x_0, x_1, \dots)$ where the x_n are played by Player I when n is even and by Player II when n is odd. At the end, Player II wins if $x \in W$, otherwise it is Player I.

Formally the game tree is simply $\omega^{<\omega}$, and the winning set is $W \subset [\omega^{<\omega}] = \omega^\omega$.

A typical play of this game looks like this:

$$\begin{array}{lcccccccccc|c} \mathtt{I} & x_0 & & & x_2 & & & x_4 & & & x_6 & \cdots & \\ & & \searrow & & \nearrow & \searrow & & \nearrow & \searrow & & \nearrow & & \rightsquigarrow x \in \omega^\omega \\ \mathtt{II} & & & x_1 & & & x_3 & & & x_5 & & \cdots & \end{array}$$

With Player I starting with an integer $x_0 \in \omega$, to which Player II answers with an integer $x_1 \in \omega$, and then back again to Player I. At each turn, each player knows all the previous moves and also know the winning condition W. The process continues for ω turns, until they produce, together a sequence $x \in \omega^\omega$. At the end of the game we only need to check whether or not x belongs to the pre-determined set W. If $x \in W$, Player II is declared the winner, otherwise, he looses and Player I wins.

An other game that is very similar to the Gale-Stewart game is the Banach-Mazur game, where, instead of playing integers, both players play sequences of integers.

Definition 3.5. Let $W \subset \omega^\omega$ be any set. The **Banach-Mazur game** of W, $BM(W)$ is the game where at each turn, the players play finite sequences of integers, $s_i \in \omega^{<\omega}$. At the end, we look at the concatenation of the sequences they produced, $x = s_0{}^\frown s_1{}^\frown s_2{}^\frown \dots$, and Player II wins if x belongs to W.

Formally, the game tree is $T = (\omega^{<\omega})^{<\omega}$ and the winning set is

$$W' = \{s \in [T] \,|\, s_0{}^\frown s_1{}^\frown \dots \in W\}.$$

Note that at any turn, players can play empty sequences and if at some point they play only empty sequences, the final concatenation will be a finite sequence. In this case, however, there is no way the concatenation belongs to W, and it will always be Player I who wins.

Those two games are at the heart of the theory of infinite games, and we will often compare new games to those simpler ones (see for instance Corollary 3.12), however we will rarely use them in this document, as we will prefer Wadge's game, which is more expressive.

Definition 3.6. Let $A, B \subset \omega^\omega$ be two sets. The **Wadge game** of A and B, $G(A, B)$ has the following rules:

- Player I chooses an integer at each turn.
- Player II chooses a finite sequences of integers, possibly empty at each turn.
- Player II cannot play empty sequences infinitely many times in a row.

After ω turns, Player I has produced ω integers that we consider as an element $a \in \omega^\omega$ and Player II has also produced a sequence $b \in \omega^\omega$ which is the concatenation of all the sequences he played.

Player II wins the game if $a \in A \iff b \in B$, otherwise Player I wins.

If we want to give a formal definition of the game, the set of moves is $X = \omega \cup \omega^{<\omega}$ and we can consider the game tree to be

$$T = \left\{ s \in X^{<\omega} \;\middle|\; \forall n \in \mathrm{lh}(s) \left(\begin{array}{c} n \text{ even} \\ \iff \\ s_n \in \omega \end{array} \right) \right\}$$

but that doesn't take into account the last condition, that Player II cannot play infinitely many empty sequences in a row. In fact this condition cannot be encoded in the game tree, because the game tree is a description of what moves are immediately possible at a each given turn, and at any given turn, Player II can play an empty sequence, he just can't do it too many times. In order to prevent that, we encode this condition in the winning set, and say that if Player II plays a finite sequence, he looses.

To simplify the notation, given an infinite play $s \in T$, we write $s^{\mathtt{I}} = \mathrm{even}(s)$ for the sequence produced by Player I and $s^{\mathtt{II}} = s_1{}^\frown s_3{}^\frown s_5{}^\frown \dots$ for the sequence produced by Player II. We can now define the winning set $W \subset [T]$ as

$$W = \left\{ s \in [T] \;\middle|\; \begin{array}{c} \mathrm{lh}(s^{\mathtt{II}}) = \omega \\ \wedge \\ s^{\mathtt{I}} \in A \iff s^{\mathtt{II}} \in B \end{array} \right\}.$$

We see that for an infinite play $s \in [T]$, Player II wins when $s \in W$ and otherwise looses is equivalent to the previous definition.

If we try to find for which sets $A, B \subset \omega^\omega$ Player $\mathtt{II}$ has a winning strategy, we find that they are precisely the sets such that there exists a continuous function $f : \omega^\omega \to \omega^\omega$ such that $A = f^{-1}(B)$. This is very useful as it means that this game allows us to understand simply what the reductions by preimages of continuous functions are, and thus study the Wadge hierarchy using games! This field has been explored in many details by Wadge in his PHD thesis of 1982 [Wadge, 1982].

Proposition 3.7. Let $A, B \subset \omega^\omega$ be two sets, Player $\mathtt{II}$ has a winning strategy in the Wadge game $G(A, B)$ if and only if there exists a continuous function $f : \omega^\omega \to \omega^\omega$ such that $A = f^{-1}(B)$.

Proof. Let τ be the winning strategy function of Player $\mathtt{II}$ in $G(A, B)$. We define the extended strategy $\tilde{\tau} : \omega^\omega \to \omega^\omega$ as the sequence produced by τ on a given input of Player $\mathtt{I}$. Formally, for $a \in \omega^\omega$, we have $\tilde{\tau}(a) = \tau(a\restriction_1)^\frown\tau(a\restriction_2)^\frown\dots$ and we claim that this function $\tilde{\tau}$ is continuous. To that extent, we only need to show that if $[t] \subset \omega^\omega$ is a basic open set, $U = \tilde{\tau}^{-1}([t])$ is open. If we take $a \in U$, a is play of Player $\mathtt{I}$ such that Player $\mathtt{II}$ produces a sequence that starts with t, however, Player $\mathtt{II}$ does this after finitely many moves (say n moves) but since the strategy is winning, it implies that regardless of how Player $\mathtt{I}$ extends his play $a\restriction_n$, Player $\mathtt{II}$ will play a sequence that starts with t and win. Thus $\tilde{\tau}([x\restriction_n]) \subset [t]$ and $\tilde{\tau}^{-1}([t])$ is indeed open.

For the other direction, we need to define a winning strategy given a function f such that $f^{-1}(B) = A$. If Player $\mathtt{II}$ manages to play $f(a)$ for any play a of Player $\mathtt{I}$, he would always win, as

$$a \in A = f^{-1}(B) \iff f(a) \in B.$$

Therefore, we need to make Player $\mathtt{II}$ "compute" $f(a)$. We define Player $\mathtt{II}$'s strategy recursively. For the base case, $\tau(\varepsilon) = \varepsilon$. For the recursive case, let $s \in \omega^{<\omega}$ be a play of Player $\mathtt{I}$, and let $t = \tau(s\restriction_1)^\frown\tau(s\restriction_2)^\frown\dots^\frown\tau(s\restriction_{\mathrm{lh}(s)-1})$ be the response of Player $\mathtt{II}$ until the previous turn. Then we set

$$\tau(s) = \begin{cases} \langle n \rangle & \text{if } f([s]) \subset [t^\frown n] \\ \varepsilon & \text{otherwise.} \end{cases}$$

This corresponds to extending Player $\mathtt{II}$'s sequence only when we are sure that regardless of the futures moves of Player $\mathtt{I}$ all possible images are share the digit n at the current position. This strategy is well defined and since the function is continuous, Player $\mathtt{II}$ cannot be in the second case infinitely many times in a row, and the sequence $\tau(a\restriction_n)$ converge indeed towards $f(a)$. □

Now that we can study Wadge reducibility with games instead of continuous functions, we can prove propositions with much less trouble.

Proposition 3.8. The set of sequences that contain an infinite amount of zeros,

$$X = \{x \in \omega^\omega \mid \exists^\infty n \; x(n) = 0\}$$

is $\boldsymbol{\Pi}^0_2$ complete. Its complement, the set of sequences that contain finitely many zeros is $\boldsymbol{\Sigma}^0_2$-complete.

Proof. First of all, $X \in \mathbf{\Pi}^0_2$ as $X = \{x \in \omega^\omega \,|\, \forall n \in \omega \; \exists m > n \; x(m) = 0\}$ and the condition $x(m) = 0$ is clearly clopen.

Let $A \in \mathbf{\Pi}^0_2$. To show that X is $\mathbf{\Pi}^0_2$-hard, we construct a winning strategy for Player `II` in the Wadge game $G(A, X)$. Since $A \in \mathbf{\Pi}^0_2$, there exists a sequence of open sets $(A_n)_{n\in\omega} \subset \mathbf{\Sigma}^0_1$ such that $A = \bigcap A_n$. Given any open set A_n, we always know in a finite amount of time if Player `I` will play a sequence $x \in A_n$, but we may not be able to tell in a finite time if $x \notin A_n$. Therefore, the strategy is defined as follows:

- Start by playing zeros.
- Until the current finite play of `I`, $s \in \omega^{<\omega}$ belongs to A_0, that is, if $[s] \subset A_0$, then play a 1
- Repeat the two steps above but with A_1 then A_2 ...

This is a winning strategy, as for an infinite play $x \in \omega^\omega$ of Player `I`, Player `II` will play a one for each n such that $x \in A_n$ until some N such that $x \notin A_N$. If this N doesn't exist, he played an infinite number of ones, and $x \in \bigcap A_n = A$, and on the other hand, if $N < \omega$, he played only N ones which is what we want since $x \notin A_N$ and therefore $x \notin A$.

In order to show that X^c is $\mathbf{\Sigma}^0_2$-complete, we can either exhibit a strategy, or directly apply Proposition 2.17. However, finding strategy always give some more insight into our sets. We therefore give a strategy to reduce X^c to any $A \in \mathbf{\Sigma}^0_2$. To that extent, we write $A = \bigcup_{n\in\omega} A_n$, with A_n a closed set. The strategy is very similar to the previous

- We start by playing zeros.
- When the current finite play of `I`, $s \in \omega^{<\omega}$ cannot end up in A_0, that is, if $[s] \subset A_0^\mathsf{c}$, then play a 1
- Then, repeat the two steps above but with A_1 then A_2 ...

This strategy is winning for the same reasons, if Player `II` plays only N zeros, it means that $x \in A_N \subset A$, but if he plays infinitely many, it is in none of the A_n and therefore not in A. □

3.3 Determinacy

In order to prove properties about the Wadge hierarchy, we will need results about the nature of the game we study, mainly about whether or not a player has a winning strategy.

Definition 3.9. We say that a game between two players is **determined** if either of the players have a winning strategy.

We already know that any finite game is determined, so it may seem at first glance that this result is also true for infinite games, say for instance a Gale-Stewart game $G(W)$. However, our intuition from finite games doesn't work here, and the proof that every finite game with perfect information is determined, does not carry to infinite games. In fact, there can be games where given any strategy τ for Player $\mathtt{I}$, Player $\mathtt{II}$ has a strategy that beats τ, and given any strategy σ for Player $\mathtt{II}$, Player $\mathtt{I}$ can find a strategy that beats σ, so none of them have a winning strategy. The problem is that the analogy with finite games fails very early, as the usual proof finite game determinacy shows that at every node of the game tree, there is one player that has a winning strategy, starting with the leaves, and backtracking all the way to the root — but there are no such things as leaves in our infinite game trees.

In fact one can still assume that all Gale-Stewart games are determined, and this assumption is called the *axiom of determinacy*, but it has some non-negligible consequences.

Definition 3.10. The **axiom of determinacy**, or **AD** is the formula that states that every Gale-Stewart game is determined.

It turns out that the **axiom of determinacy** is very strong. For instance, it can be proved that it implies that every subset of the reals is measurable, and thus that it contradicts the axiom of choice. Woodin also proved that **ZF** + **AD** is equiconsistent with **ZFC** together with the existence of infinitely many Woodin cardinals. That is, if **AD** holds, then it also holds that there are infinitely many inaccessible cardinals.

However, most of the time we do not need the full axiom of determinacy as we only need to use the determinacy of the games that we use, and since we mostly work with Borel sets, we only need the determinacy of Borel games.

Theorem 3.11 (Martin 75)**.** In **ZF** + **DC**, every Borel game is determinate. That is, if $W \subset \omega^\omega$ is a Borel set, then the Gale-Stewart game $G(W)$ is determined.

Even if this theorem do not apply directly to Wadge's game, it implies the determinacy of Wadge's game between two Borel sets.

Corollary 3.12. Let $A, B \subset \omega^\omega$ be two Borel sets. Under **ZF** + **DC** the Wadge game $G(A, B)$ is determined.

Proof. Given A and B two Borel sets, it suffices to transform the game $G(A, B)$ into a Gale-Stewart game $G(W)$ for some Borel set W, in such a way that we can use the winning strategy in $G(W)$ provided by Theorem 3.11 to compute a winning strategy in $G(A, B)$.

Without loss of generality, we can assume that in the Wadge game $G(A, B)$, Player $\mathtt{II}$ can play at most one integer at each turn. This doesn't change whether any player has a winning strategy, since a winning strategy for Player I would still work by restricting Player II moves, and on the other hand, if Player I doesn't have a winning strategy, he cannot have one if Player II gives him less information.

This means that the Wadge game is equivalent to the game where Player I plays integers and Player II plays either an integer or a token S which is interpreted as skipping his turn, that is, playing the empty sequence. We can encode moves of Player II only with integers, for instance with the mapping $f : \omega \cup \{\mathtt{S}\} \to \omega$ defined as

$$f(n) = \begin{cases} 0 & \text{if } x = \mathtt{S} \\ n+1 & \text{otherwise.} \end{cases}$$

In this "translation" of the game, that we call G', each player now plays integers in turn, resulting in one sequence of integers, $x \in \omega^\omega$, therefore, it can be seen as a Gale-Stewart game.

We define the winning set in G' to correspond to the winning condition of the Wadge game, that is on an infinite play $x \in \omega^\omega$, Player II wins if and only if

- $\exists^\infty n \in \omega\ x_{2n+1} \neq 0$, that is, Player II plays infinitely many non empty sequences, which are represented by zeros.
- $x^{\mathtt{I}} \in A \iff x^{\mathtt{II}} \in B$, where $x^{\mathtt{I}} = \mathrm{even}(x)$ is the sequence of Player I and $x^{\mathtt{II}} = \mathrm{strip}_0(\mathrm{odd}(x)) - 1$ is the decoded play of Player II, where
  - $\mathrm{strip}_0 : \omega^\omega \to \omega^{\leq\omega}$ is the function that associates to a sequence the same sequence but without the zeros, that is, without the "skipped turns". Note that this function, restricted to the preimage of ω^ω is a continuous function.
  - the -1 is done on each digit of the sequence.

The desired winning set is therefore

$$W = \left\{ x \in \omega^\omega \;\middle|\; \exists^\infty n \in \omega\ \ \mathrm{odd}(x)(n) \neq 0 \wedge \left(\begin{array}{c} \mathrm{even}(x) \in A \\ \iff \\ \mathrm{strip}_0(\mathrm{odd}(x)) - 1 \in B \end{array} \right) \right\}.$$

But we can also write in a form that shows that it is a Borel set, provided that A and B are Borel sets.

$$W = P \cap (A \mathbin{\S} B)$$

where

- $P = \mathrm{odd}^{-1}(\mathrm{strip}_0^{-1}(\omega^\omega)) = \{x \in \omega^\omega \mid \exists^\infty n \in \omega\ x_{2n+1} \neq 0\}$ is the set of sequence with infinitely many non-zero digits at odd places. P is $\mathbf{\Pi}^0_2$.
- $B' = \mathrm{strip}_0^{-1}(f^{-1}(B))$ is the set of sequences that are mapped to B when decoded. B' is Borel as the preimage of a Borel set by continuous functions.
- $A \mathbin{\S} B' = \{x \in \omega^\omega \mid \mathrm{even}(s) \in A \wedge \mathrm{odd}(s) \in B'\}$ is the set of sequences of A interleaved with sequences of B'. $A \mathbin{\S} B'$ is Borelian as it is the preimage of $A \times B'$ by the isomorphism $\omega^\omega \times \omega^\omega \to \omega^\omega$ that consist of interleaving sequences.

Therefore W is Borel, so we know that there is a winning strategy for one of the players in G' by Theorem 3.11. This strategy can be translated to $G(A, B)$ by translating moves back and forth. That is, if it is a winning strategy for Player I, Player I starts by playing the same integer as the strategy, then looks at what Player II plays in $G(A, B)$, translates it to a play in the game G', and plays what its winning strategy tells in G'. The same goes if it is a winning strategy for Player II. □

Now that we have those determinacy results, we can make use of arguments such as "If Player II doesn't have a winning strategy, then Player I has one".

4 The Tree game

In the next section, our aim will be to construct a game-theoretic characterisation of Borel functions. A Borel function is a function such that the preimage of an open set is always a Borel set. Given a function $f : \omega^\omega \to \omega^\omega$, we will construct a game $G(f)$ such that Player II has a winning strategy whenever f is a Borel function.

This game is due to [Semmes, 2009] and we will reproduce here his construction and proof, however with greater detail.

In Semmes' game, at each turn Player I plays an integer x_n, similarly to Wadge's game. In respond to this, Player II answers with a finite colored tree $C_n : T_n \to \omega^{<\omega}$, with T_n being any finite tree. At each step $n > 1$, the new colored tree C_n must extend the previous tree C_{n-1}, so that Player II builds up a colored tree, one step at a time. Some confusion can arise from the fact that there are many different trees involved, since C_n is a colored tree, that maps a finite tree T_n into an other tree $C_n(T_n) \in \omega^{<\omega}$. It is important to have those different trees, and the map C_n allow to "merge" branches of T_n into the same branch inside $\omega^{<\omega}$. This will prove necessary, as we will see in the example of subsection 4.2.

At the end of the game, we say that Player I *produces* the sequence $x \in \omega^\omega$ and Player II *produces* the colored tree $C = \bigcup C_n$. Player II wins the game if $\mathrm{dom}(C)$ has a unique infinite branch z and

$$\bigcup_{s \subset z} C(s) = f(x)$$

that is, the color y along z is precisely the image of x. Otherwise, Player I wins.

More formally, the game is defined as follows.

Definition 4.1. Given a function $f : \omega^\omega \to \omega^\omega$, the **tree game** of f, is the game $G(f) = (T, W)$, where the game tree T is the set of all finite sequences s such that:

- $\mathrm{odd}(s) \in \omega^{<\omega}$, i.e. Player I constructs a sequence of integers.
- If we write $C = \mathrm{even}(s)$ for the moves of Player II, then for all n, $C_n : T_n \to \omega^{<\omega}$ is a finite colored tree, with $T_n \subset \omega^{<\omega}$ such that the coloring map is:
  - length preserving, so for all $u \in \mathrm{dom}(C_n)$, $\mathrm{lh}(u) = \mathrm{lh}(C_n(u))$.

- monotone, so if $u \subset t \implies C_n(u) \subset t \implies C_n(t)$.
- is extended by the next move $C_n \subset C_{n+1}$.

For a given play $s \in [T]$, let $s^{\mathtt{I}} = \bigcup \mathrm{odd}(s) \in \omega^\omega$ be the sequence produced by Player I, and $s^{\mathtt{II}} = \bigcup \mathrm{even}(s)$ be the colored tree produced by Player II. $s^{\mathtt{II}}$ is a map from a tree $T \subset \omega^{<\omega}$ to $\omega^{<\omega}$.

The winning set is

$$W = \left\{ s \in [T] \,\middle|\, \exists z \in \omega^\omega \begin{pmatrix} \left[\mathrm{dom}\left(s^{\mathtt{II}}\right)\right] = \{z\} \\ \wedge \\ f(s^{\mathtt{I}}) = \bigcup_{n\in\omega} s^{\mathtt{II}}(z\restriction_n) \end{pmatrix} \right\}$$

where $\left[\mathrm{dom}\left(s^{\mathtt{II}}\right)\right]$ is the set of infinite branches of the colored tree constructed by Player II and z is its unique infinite branch. The second condition ensures that the color along z is $f(x)$.

A Borel function is a function f such that for every open set $U \subset \omega^\omega$, $f^{-1}(U)$ is a Borel set. We will show in Theorem 4.3 that the functions for which Player II has a winning strategy are precisely the Borel functions.

4.1 Continuous functions

We will see that, if $f : \omega^\omega \to \omega^\omega$ is a continuous function, then Player II has a winning strategy in $G(f)$. In fact even if we make the rules much harder for Player II, that is, if he is allowed to to play colored trees that have only one branch, he still has a winning strategy. We prove the following:

Theorem 4.2. A function $f : \omega^\omega \to \omega^\omega$ is continuous if and only if Player II has a winning strategy in $G(f)$ that involves a unique branch.

Since this strategy involves a tree with a single branch, we will consider that Player II plays a sequence. With this view point, let $x_n \in \omega^{<\omega}$ the sequence played by Player I on turn n and $y_n \in \omega^{<\omega}$ the sequence played by Player II on that turns. The rules translate as follows:

- $x_0 = \langle k \rangle$ for $k \in \omega$ and $x_{n+1} = x_n{}^\frown k$ for $k \in \omega$, so Player I constructs a sequence one digit at a time.
- $y_0 \in \omega^{<\omega}$ and $y_{n+1} = y_n{}^\frown s$ for $s \in \omega^{<\omega}$, so Player II constructs a sequence, many steps at a time, and potentially zero.

Player II wins if his sequence $y = \bigcup y_n$ is infinite and $f(x) = y$.

A winning strategy for Player II Let $f : \omega^\omega \to \omega^\omega$ be continuous function. We will build a winning strategy function τ for Player $\mathtt{II}$. Let s be the current play of Player $\mathtt{I}$. We set

$$X_s = \left\{ t \in \omega^{<\omega} \,\middle|\, f([s]) \subset [t] \right\}$$

to be the set of all sequences that are a prefix of the image of any sequence that Player $\mathtt{I}$ can still play. We notice that X_s is naturally well ordered, since if we have two finite sequences $t, t' \in X_s$, then either $t \subset t'$ or $t' \subset t$ since both are a prefix of all sequences in $f([x])$. This set is also non-empty, since $\varepsilon \in X_s$. There are two cases:

- Either X_s is infinite and that would mean that $f{\restriction_{[s]}}$ is constant. Let y be the image of f on $[s]$, we can take $\tau(s) = y{\restriction_{\mathrm{lh}(s)}}$, to make sure that we build an infinite sequence that converges to y.

- Otherwise, we set $\tau(s)$ to be the longest sequence of X_s. This is always a valid move, since for any $k \in \omega$, $f([s^\frown k]) \subset f([s]) \subset [t]$ and thus $X_s \subset X_{s^\frown k}$.

Now, this strategy is winning, as for any $x \in \omega^\omega$, $n \in \omega$, and $y = f(x)$, we have that $U := f^{-1}([y{\restriction_n}])$ is open since f is continuous, but $x \in U$ and therefore there must exist $m \in \omega$ such that $[x{\restriction_m}] \subset U$. This means that on turn m, Player $\mathtt{II}$ will have played $y{\restriction_n}$ and since we have already shown that he plays only valid moves, the sequence $\bigcup_{n\in\omega} \tau(s{\restriction_n}) = \bigcup y_n$ played by Player $\mathtt{II}$ is precisely y.

A winning strategy defines a continuous function. To conclude, we need to show that if Player $\mathtt{II}$ has a winning strategy τ in $G(f)$, then f is continuous. It suffices to show that for $t \in \omega^{<\omega}$, $U_t := f^{-1}([t])$ is open. U_t is the set of all plays of Player $\mathtt{I}$ such that, at some point, Player $\mathtt{II}$ plays the sequence t. If U_t is non-empty, let $x \in U_t$. We know that at some point $m \in \omega$, $t \subset \tau(x{\restriction_m})$ because the strategy is winning, so Player $\mathtt{II}$ has to play t. However, since τ is a well defined strategy, for all finite sequences that extends $x{\restriction_m}$, Player $\mathtt{II}$ also play t therefore, $[x{\restriction_m}] \subset U_t$. This finishes the proof that U_t is open and thus that this modified game describes precisely the continuous functions.

4.2 Counting zeros

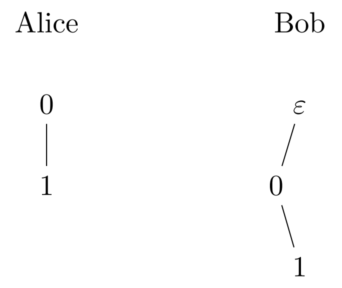
We first start by looking at an example to motivate the definition of our game. We define the *counting zeros* function as

$$\begin{aligned} f :\ \omega^{<\omega} &\longrightarrow \omega^{<\omega} \\ x &\longmapsto \begin{cases} \underbrace{0\dots0}_{k \text{ times}}111\dots & \text{if } x \text{ contains exactly } k \text{ zeros} \\ 1111\dots & \text{if } x \text{ has infinitely many zeros.} \end{cases} \end{aligned}$$

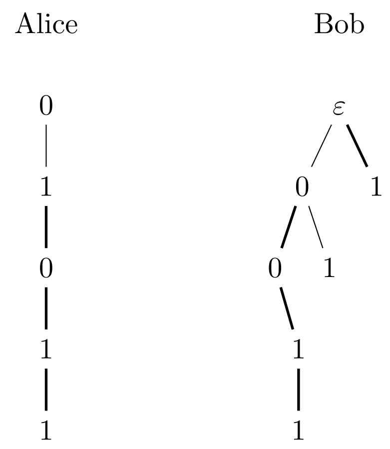
We'll call our players Alice (for Player $\mathtt{I}$, pronoun: she) and Bob (for Player $\mathtt{II}$, pronoun: he).

The biggest challenge for Bob is that at any finite moment, he doesn't know whether Alice will keep playing ones and be in the first branch of the definition, or if she will always play some zeros again.

If we make the consider that Bob plays trees instead of colored tree[4], a sample play could look like the following, where we represent only the sequence that Alice is building and Bob's tree.



For the first two moves, Alice plays 0 then 1, and so does Bob, as potentially Alice can now play only ones and if it is the case, Bob needs to play only ones too. So Bob's strategy is to continue to play ones as long as Alice does.



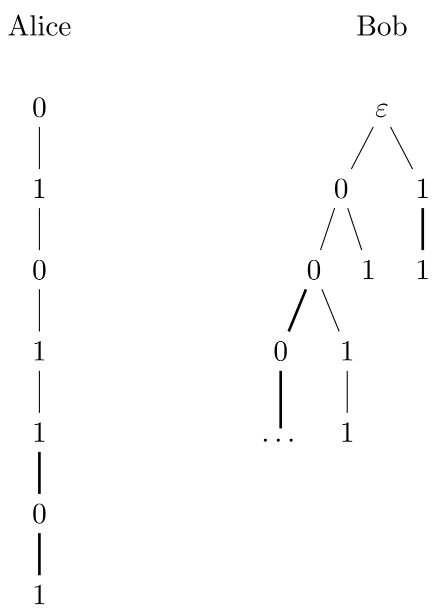
But on the next turn, Alice added a 0, and then two ones (new moves are in bold). If Alice continues to play ones, Bob will need a branch that reads $(0, 0, 1, 1, \dots)$ so he chooses to play it straight away.

However, if Alice continues to play zeros, he needs to have a branch that reads $(1, 1, 1, \dots)$, so Bob decides that every time Alice plays a 0, he'll add a 1 to this branch.

In fact, in this game Alice can force Bob to have a branch that starts with n zeros, as if she has n zeros at some point in her sequence and start playing only ones, Bob can win only if he has a branch that starts with n zeros and then ones.

That means that Alice can actually force Bob into playing an infinite branch consisting of zeros! She just forces him to play n zeros, then $n+1 \dots$ At this point she would

[4] Notice that a tree colored with the identity function is a colored tree that satisfies all the conditions of the game.



have won, since Bob cannot have at the same time and infinite branch of zeros and the correct image of the sequence Alice played, which should be $(1, 1, \dots)$.

This illustrates why colored trees are necessary, as Bob *has* a strategy with colored trees. Every time he wants to play a branch starting with n zeros and then only ones, it can be disjoint, as the domain of the colored tree can be anything. In particular, only the color along a branch is taken into account, be not the values of the infinite branch of the domain itself.

Here is an example of what Bob would have played with colored trees. We represent the function $C_i : T_i \to \omega^{<\omega}$ by writing the branches of T_i as indices, and the color along this branch is read in bold numbers.

To define formally the strategy, let

$$S = \left\{ s \in \omega^{<\omega} \,\middle|\, \forall i \in \omega \ (0 < i < \mathrm{lh}(s) \implies s_i = 0) \right\}$$

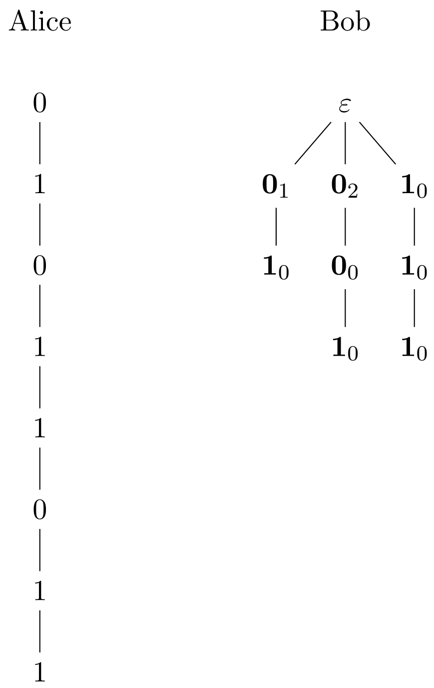
be the set off all finite sequences that have only zeros except maybe for their first element. Also, let

$$\begin{array}{rrcl} \varphi : & S & \longrightarrow & \omega^{<\omega} \\ & s & \longmapsto & \begin{cases} \varepsilon & \text{if } s = \varepsilon \\ (\underbrace{0, \dots, 0}_{n \text{ times}}, 1, \dots)\restriction_k & \text{if } s = (n, \underbrace{0, \dots, 0}_{k-1 \text{ times}}) \end{cases} \end{array}$$

be a length preserving, monotone function.

A winning strategy for Bob in $G(f)$ can be described by induction as follows:

- Start with $C_0 : \varnothing \to \omega^{<\omega}$ the empty function.



- On turn $n+1$, if Alice plays her k-th zero, play $C_{n+1} = C_n \cup \varphi\restriction_{X_k}$, where $X_k = \{0\}^{<k}$ is the set of sequences of length less than k consisting only of zeros.
- On turn $n+1$, if Alice plays a non zero number after a total of exactly k zeros, play $C_{n+1} = C_n \cup \varphi\restriction_{X_{k,n}}$, where $X_{k,n} = k^\frown\{0\}^{<n}$ is the set of sequences of length less than n that starts with a k and the only zeros.

At any turn, $C_n \subset \varphi$ and $\mathrm{dom}(C_n)$ is a tree by construction, so our strategy is indeed a strategy. To show that it is winning, we only need to check that on input $x \in \omega^\omega$ from Player `I`, the strategy produces a unique infinite branch along which the color is $f(x)$. Let $C \cup C_n$ be the colored tree produced by Player `II`.

If x has exactly k zeros, after a finite amount of time, we will always be in the third case of the strategy, so the only infinite branch of C will be $(k, 0, 0, \dots)$ and the color along this branch is $(\underbrace{0, \dots, 0}_{k \text{ times}}, 1, 1, \dots) = f(x)$

If x has infinitely many zeros, we hit infinitely many times the second case, and thus $(0, 0, \dots) \in \mathrm{dom}(C)$. Since the color along this branch is $(1, 1, \dots) = f(x)$ we need to show that it is the only infinite branch, but the infinite branches of $\mathrm{dom}(\varphi)$ are of the form $b_k = (k, 0, 0, \dots)$ thus it suffices to show that for a given $k > 0$, $b_k \notin \mathrm{dom}(C)$. This is the case since Bob extends prefixes of b_k only when Alice plays a non zero number after exactly k zeros and this can happen only finitely many times since the number of zeros goes to infinity.

4.3 Baire functions

Theorem 4.3. Let $f : \omega^\omega \to \omega^\omega$ be a function. Player II has a winning strategy in the tree game $G(f)$ if and only if f is a Borel function.

The proof of this theorem is quite long, and will be split in multiple parts:

1. We prove that if Player II has a winning strategy then for all basic open set $[s]$, $f^{-1}([s])$ is analytic and co-analytic. By Theorem 2.28, it a Borel set and thus f is a Borel function.

2. For the other direction, we will show that the set of functions where Player II has a strategy is closed under pointwise limits and since it contains the continuous functions, it must contains all the Borel functions.

Proof of $\implies$ Let $f : \omega^\omega \to \omega^\omega$ be a function such that Player II has a winning strategy in $G(f)$. We denote the strategy function by τ.

Let $[t] \subset \omega^\omega$ be a basic open set. We first show that $A_t = f^{-1}([t])$ is analytic. We have that A_t is the set of all reals x such that Player II produces a sequence that starts with t. Formally,

$$A_t = \left\{ x \in \omega^\omega \;\middle|\; \exists z \in \omega^\omega \begin{pmatrix} \exists m \in \omega \; \tau(x\restriction_m)(z\restriction_{\mathrm{lh}(t)}) = t \\ \wedge \\ \forall n \in \omega \; \exists m \in \omega \; z\restriction_n \in \mathrm{dom}(\tau(x\restriction_m)) \end{pmatrix} \right\}$$

Here, z is the unique infinite branch of the colored tree produced by Player II. The first condition ensures that at some turn m the color of the initial segment of z is precisely t. Since Player II only extends his colored tree it would also be the case for all subsequent turns, so the sequence played by II indeed starts with t. The second condition ensures that z is an infinite branch of the colored tree, because for any prefix of z of length n, there is some turn m where it is part of the tree.

Now for a fixed m, n and z, both conditions depend only on $x\restriction_m$ and are thus clopen conditions. Every time we have a quantifier $\forall X \in \omega$ or $\exists X \in \omega$ it is equivalent to a countable intersection or a countable union respectively, so the set of couples (x, z) that satisfy it is a Borel set. By Proposition 2.25, A_t is an analytic set since it is the projection along a coordinate of a Borel set.

Proof of $\impliedby$ The proof in the other direction is a bit more involved. We need to show that for any Borel function Player II has a winning strategy. Let $\mathcal{F}$ be the set of functions such that Player II has a winning strategy.

In our first example of the tree game, we have seen that continuous functions have a winning strategy and thus belong to $\mathcal{F}$. If we show that $\mathcal{F}$ is closed under pointwise limits, we'll have that all Borel functions are contained in $\mathcal{F}$, since the set of Borel functions is the smallest set that contains the continuous functions and is closed under pointwise limits.

To show the closure property, take a sequence $f_n \in \mathcal{F}$ such that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in \omega^\omega$. We have winning strategies T_n for each f_n and we need to combine them in a strategy for f. There are two convergences that interest us:

- For a given n and a play x of Player I, we note $z_{n,x}$ the unique infinite branch of the colored tree produced T_n in the game $G(f_n)$. We don't know the those sequences ahead of time, but the longest branch at each turn will converge towards it.

- We would also like to know the rate of convergence of the $f_n(x)$ towards y. By rate of convergence, we mean the sequence $r_x \in \omega^\omega$ such that $N = r_x(m)$ is the least number for which

$$\forall n > N,\ f_n(x){\restriction_m} = f_N(x)\restriction_m .$$

  This N is the first index for which the m first integers of the images will not change for larger indices.

It would be easy to compute $f(x)$ if we knew at any given point in the game how those convergences behave. Since we don't know them, we will make finite guesses for both the $z_{n,x}$ and r_x that will allow us to derive a winning strategy for f.

The guessing functions We define two guessing functions

$$\gamma_0 : \omega^{<\omega} \to \omega \quad \text{and} \quad \gamma_1 : \omega^{<\omega} \to (\omega^{<\omega})^{<\omega}.$$

Each of these functions takes as an input sequence s that will only serve to index our guesses. $\gamma_0(s)$ will be our guess for $r_x(\mathrm{lh}(s))$, so we think that for n greater than $\gamma_0(s)$ all the f_n agree on the first $\mathrm{lh}(s)$ integers. On the other hand, $\gamma_1(s)$ is a list of guesses for the $y_{n,x}$, and more precisely $\gamma_1(s)(i)$ will be our guess for $z_{i,x}\restriction_{\mathrm{lh}(s)}$, i.e. the first $\mathrm{lh}(s)$ integers of the infinite branch produced on the i-th function.

The definition of the guessing functions are done by induction. For the base case, the empty sequence ε, we set $\gamma_0(\varepsilon) = 0$ and $\gamma_1(\varepsilon) = \langle\varepsilon\rangle$. For the recursive case, we assume that $\gamma_0(s) = N$ and $\gamma_1(s) = \langle s_0, s_1, \ldots, s_k\rangle$ are defined and we make the pairs $(\gamma_0(s^\frown j), \gamma_1(s^\frown j))$ enumerate all the pairs

$$(N', \langle u_0, u_1, \ldots, u_{k'}\rangle)$$

such that:

- $N' \geq N$, as the f_n converge on the first $\mathrm{lh}(s^\frown j)$ digits later than on the first $\mathrm{lh}(s)$.

- $k' = \max(N', \mathrm{lh}(s)) + 1$. For technical reasons, as we both want $\gamma_0(s)$ to be in the domain of $\gamma_1(s)$ and at the same time that $\lim_{s\to z}\gamma_1(s) = \infty$. This way we make sure that we will get guesses of all the $z_{n,x}$.

- $\mathrm{lh}(u_i) = \mathrm{lh}(s)+1$, as we always guess the same number of values as our index (here $s^\frown j$).

- for $i \leq k$, our new guess extends the previous one: $u_i\restriction_{\mathrm{lh}(s_i)} = s_i$.

Proposition 4.4. For $s, t \in \omega^{<\omega}$, this definition of guessing functions satisfy the following simple properties:

1. $s \subset t \implies \gamma_0(s) \leq \gamma_0(u)$, so γ_0 is increasing along any sequence.

2. $\mathrm{lh}(\gamma_1(s)) = \max(\gamma_0(s), \mathrm{lh}(s)) + 1$.

3. for all $i < \mathrm{lh}(\gamma_1(s))$, we have $\mathrm{lh}(\gamma_1(s)(i)) = \mathrm{lh}(s)$ so all guessed sequences have the same length as the indexing sequence.

4. $s \subset t \implies \gamma_1(s)(i) \subset \gamma_1(t)(i)$, for $i < \mathrm{lh}(\gamma_1(s))$, the guesses extends previous guesses.

But the important property is that for any every non-decreasing $r \in \omega^\omega$ and any sequence $z_n \in \omega^\omega$ is described by a $z \in \omega^\omega$ via γ_0 and γ_1. On the other hand, every $z \in \omega^\omega$ describes one such r and z_n. This means that our guesses effectively encode all the possibilities for the convergences.

Active sequences To define Player `II`'s strategy we'll use the concept of an *active sequence*. An active sequence is a guess that might be correct considering the information we have from Player `I` and our previous guesses.

Formally, let $p \in \omega^{<\omega}$ be a finite play of Player `I`, $s \in \omega^{<\omega}$ be an index sequence and $L = \mathrm{lh}(\gamma_1(s))$ the number of guesses corresponding to s. For $i < L$, we write $s_i = \gamma_1(s)(i)$ for our guess of the correct branch in the i-th strategy and $t_i = \tau_i(p)(s_i)$ for the color of this guess by the strategy τ_i.

We say that s is **active** if:

- for all $i < L$ we have $s_i \in \mathrm{dom}(\tau_i(p))$, so each guess is in the domain of the colored tree played by Player `II` in the game of f_i. When some $s_i \notin \mathrm{dom}(\tau_i(p))$ then we are not interested in this guess (yet). This ensure that when s is an active sequence, $t_i := \tau_i(p)(s_i)$ is well defined.

- for all $m \leq \mathrm{lh}(s)$ such that $N := \gamma_0(s\restriction_m) > 0$ we have $t_N\restriction_m \neq t_{N-1}\restriction_m$. N is our guess for $r_x(m)$, that is, we assume that the functions converge on the first m digits precisely at the N-th function, so if $t_N\restriction_m = t_{N-1}\restriction_m$ then our guess was too big.

- for all $m \leq \mathrm{lh}(s)$ and for all n such that $N < n < \mathrm{lh}(\gamma_1(s))$ we have $t_N\restriction_m = t_n\restriction_m$. This is similar to the previous condition, but ensures that our guess for the convergence wasn't too small, i.e. the functions converge on the m first digits only later.

The strategy To define the strategy function, we need a finite set of active sequence

$$S(p) := \left\{ s \in \omega^{<\omega} \,\middle|\, s \text{ is active and } \mathrm{lh}(\gamma_1(s)) \leq \mathrm{lh}(p) \right\}.$$

Recall that for $p \in \omega^{<\omega}$ from Player $\mathtt{I}$ and s active

$$t_i = \tau_i(p)(\gamma_1(s)(i)).$$

Finally, we set $\tau(p)$ to be the function

$$\begin{array}{rccc} \varphi: & S(p) & \longrightarrow & \omega^{<\omega} \\ & s & \longmapsto & t_{\gamma_0(s)}. \end{array}$$

This φ is in fact a colored tree on the active sequences that map an active sequence s to the guess $t_{\gamma_0(s)}$. This guess can be seen as the most precise guess that can be made according to $\gamma_0(s)$ and $\gamma_1(s)$.

To show that τ is a winning strategy we first prove that it is a strategy with the three following claims.

Claim. $\mathrm{dom}(\tau(p))$ is a tree.

Proof. It suffices to show that if we have two sequences $s \subset u$ and u is active, then s is also active. We check the three conditions:

- Let $i < \mathrm{lh}(\gamma_1(s))$. Since $i < \mathrm{lh}(\gamma_1(s)) \leq \mathrm{lh}(\gamma_1(u))$, and u is active, we have $u_i := \gamma_i(u)(i) \in \mathrm{dom}(\tau_i(p))$. Then, by Proposition 4.4, $s_i := \gamma_i(s)(i) \subset u_i$. Since $\mathrm{dom}(\tau_i(p))$ is a tree, we indeed have $s_i \in \mathrm{dom}(\tau_i(p))$.
- Let $m \leq \mathrm{lh}(s)$ and such that $N := \gamma_0(s{\restriction}_m) > 0$. By the second condition of activation of u, $u_N{\restriction}_m \neq u_{N-1}{\restriction}_m$. Again by Proposition 4.4, we have $s_i \subset u_i$ for all $i < \mathrm{lh}(\gamma_1(s))$ and thus $s_N{\restriction}_m \neq s_{N-1}{\restriction}_m$.
- Similarly, let $m \leq \mathrm{lh}(s)$ and for all n such that $N < n < \mathrm{lh}(\gamma_1(s))$, by the third condition of activation of u, $u_N{\restriction}_m = u_n{\restriction}_m$, but since for $i < \mathrm{lh}(\gamma_1(s))$ we have $s_i{\restriction}_m = u_i{\restriction}_m$ we conclude that s is active.

This shows that $\mathrm{dom}(\tau(p))$ is a tree. □

Claim. $\mathrm{dom}(\tau(p))$ is finite.

Proof. Let $p \in \omega^{<\omega}$ and $k \in \omega$. It suffices to show that

$$S_k(p) = \left\{ s \in \omega^{<\omega} \,\middle|\, s \text{ is active and } \mathrm{lh}(\gamma_1(s)) = k \right\}$$

is finite. First we notice that k is an upper bound of $\gamma_0(s)$, by the second point of Proposition 4.4. By the first condition of active, there are finitely many possibilities for $\gamma_1(s)$ since $\mathrm{dom}(\tau_i(p))$ is finite and $\gamma_1(s)(i) \in \mathrm{dom}\,\tau_i(p)$. Also, for a given $n \in \omega$ and $\langle u_0, \ldots, u_k \rangle \in \omega^{<\omega^{<\omega}}$ there are finitely many s such that $\gamma_0(s) = n$ and $\gamma_1(s) = \langle u_0, \ldots, u_k \rangle$, because in your construction of the guessing sequences, it would correspond to different ways of extending our guesses to reach $(\gamma_0(s), \gamma_1(s))$. □

Claim. The map $\tau(p)$ is length preserving, monotone and if $p \subset q$ are two plays, then $\tau(p) \subset \tau(q)$.

Proof. First, $\tau(p)$ is length preserving since for a given $s \in \omega^{<\omega}$ our guesses all have the same length as s and τ_i is also length preserving, so $\mathrm{lh}(s) = \mathrm{lh}(\gamma_1(s)(\gamma_0(s))) = \mathrm{lh}(t_{\gamma_0(s)})$.

To show that $\tau(p)$ is monotone, let $s \subset u \in \mathrm{dom}(\tau(p))$ be two sequences. We need to show that $\tau(p)(s) \subset \tau(p)(u)$. One last time, we set $u_i = \tau_i(p)(\gamma_1(u)(i))$ and $s_i = \tau_i(p)(\gamma_1(s)(i))$ and we have

$$\tau(p)(s) = s_{\gamma_0(s)} \underset{\substack{\uparrow \\ \text{Prop. 4.4}}}{=} u_{\gamma_0(s)}\restriction_{\mathrm{lh}(s)} \underset{\substack{\uparrow \\ (*)}}{=} u_{\gamma_0(u)}\restriction_{\mathrm{lh}(s)} \subset \tau(p)(u).$$

For $(*)$, we need to consider the third condition for the activity of u, with $m = \mathrm{lh}(s)$ and $n = \gamma_0(u)$ since that gives us exactly $u_{\gamma_0(s\restriction_{\mathrm{lh}(s)})}\restriction_m = u_{\gamma_0(u)}\restriction_m$.

Finally, if $p \subset q$, then for all $i \in \omega$, $\tau_i(p) \subset \tau_i(q)$ and we need to show that $\tau(p) \subset \tau(q)$. To that end, note that $S(p) \subset S(q)$ as an active sequence stays active later in the game, since the definition of active doesn't depend on it. If $s \in S(p)$, we can conclude by showing that $\tau(p)(s) = \tau(q)(s)$. If we write $\tilde{s} = \gamma_1(s)(\gamma_0(s))$:

$$\tau(p)(s) = \tau_{\gamma_0(s)}(p)(\tilde{s}) = \tau_{\gamma_0(s)}(q)(\tilde{s}) = \tau(q)(s).$$

□

Those three claims combined ensure that the moves $\tau(p)$ are valid, and thus τ is a strategy.

The strategy is winning It remains to be shown that τ is in fact a winning strategy, that is, on any input x, τ produces a unique infinite branch along which the value is $f(x)$.

Let r_x be the rate of convergence of the f_n and $z_{n,x}$ be the unique infinite branches produced by τ_n on input x, then let $z \in \omega^\omega$ be the unique sequence such that for all finite sequence $s \subset z$,

$$\gamma_0(s) = r_x(\mathrm{lh}(s)) \quad \text{and} \quad \gamma_1(s) = \langle s_0, \ldots, s_k \rangle$$

and $s_i \subset z_{n,x}$. This z is the unique sequence that encode all the correct guesses via γ_0 and γ_1 for the game. Now let φ_x be the colored tree produced by Player `II`. To show that z is an infinite branch of φ_x, we can see that, since any $s \subsetneq z$ makes only correct guesses, it will be active at some point, so $s \in \varphi_x$ and since $\gamma_0(s)$ is exactly the number of digits that have converged, $\varphi_x(s) = f(x)\restriction_{\mathrm{lh}(s)}$.

The very last thing to show is that z is the only infinite branch of $\mathrm{dom}(\varphi_x)$. To this extent, suppose that there exists an other infinite branch $z' \neq z$. We will show that there is an initial segment of z' that is never activated. Let z'_n the infinite branches encoded by z' via γ_1 and let $\varphi_{n,x}$ the colored trees produced by τ_n.

There are two cases. Either $z'_n \neq z_n$ for some $n \in \omega$ and then there is an initial segment $s \subset z'$ such that $s \notin \varphi_{n,x}$ otherwise τ_n would produce two infinite branches. Let

$u \subset z$ be any sequence such that $s \subset \gamma_1(u)(n)$, then u cannot be activated as it would contradict the first condition of activation. Contradiction.

On the other hand, if for all $n \in \omega$ we suppose that $z'_n = z_n$, then there must be some $s \subset z'$ such that $\gamma_0(s) \neq r_x(\mathrm{lh}(s))$, otherwise the two sequences z and z' would encode the same guesses which is impossible by construction. If $\gamma_0(s) > r_x(\mathrm{lh}(s))$, then s is never activated because it is a guess for convergence $\gamma_0(s)$ that is too big and the second condition of activation will not be met. If $\gamma_0(s) < r_x(\mathrm{lh}(s))$, we need to find the sequence that witness that this guess is too small, i.e. that the first index starting from which the f_n agree on the first $\mathrm{lh}(s)$ digits is is not $\gamma_0(s)$ but bigger. Formally, there must be an $i > \gamma_0(s)$ such that $f_{\gamma_0(s)}(x)\restriction_{\mathrm{lh}(s)} \neq f_i(x)\restriction_{\mathrm{lh}(s)}$. We pick any $u \in \omega^{<\omega}$ such that $s \subset u \subset z'$ and $\gamma_1(u) = \langle u_0, \ldots, u_k \rangle$ with $i < k$. Then this u cannot be activated since the u_i witness that the guess $\gamma_0(s)$ is too small and this violated the third condition of activation.

This concludes the proof that τ is a winning strategy and therefore we have shown Theorem 4.3. □

4.4 A game for $\mathbf{\Lambda}_{2,2}$ functions

The game for continuous function was quite simple, and Player `II` had only one liberty: he could pass its turn, which corresponded to building a tree with a single branch. In other games, we will give more power to Player `II`, but not as much power as for the tree game.

Imagine that in the game of continuous functions, we give Player `II` an big eraser, so that he can at any point, erase the whole sequence he produced and start over. Of course still want him to produce and infinite sequence, so he can't erase everything an infinite number of times.

We will show that in this new game, Player `II` has a winning strategy if and only if the function is $\mathbf{\Lambda}_{2,2}$.

Definition 4.5. Let $f : \omega^\omega \to \omega^\omega$ be a function. The **backtrack game**, $G_{\Uparrow}(f)$ of f where $\Uparrow$ represent a big eraser is defined as follows:

- The alphabet of moves is $X = \omega \cup \{\Uparrow\} \cup \omega^{<\omega}$.
- At each turn, Player `I` plays an integer $x_n \in \omega$, then Player `II` plays a finite sequence of integers or $\Uparrow$, so that the game tree is:

$$T = \left\{ s \in X^\omega \;\middle|\; \begin{array}{c} \mathrm{even}(s) \in \omega^\omega \\ \wedge \\ \forall n \in \omega \;\; \mathrm{odd}(s)(n) \in \{\Uparrow\} \cup \omega^{<\omega} \end{array} \right\}.$$

- For an infinite play $s \in T$ we say that Player `I` produces $s^{\mathtt{I}} = \mathrm{even}(s)$.

- To define what Player $\texttt{II}$ produces, let $k = \sup \{n \in \omega \mid \mathrm{odd}(s)(n) = \Uparrow\} \cup \{-1\}$. If $k = \omega$, an infinite amount of $\Uparrow$ have been played, we set $s^{\texttt{II}} = \varepsilon$ the empty sequence. Otherwise, $s^{\texttt{II}} = \mathrm{odd}(s)(k+1)^\frown \mathrm{odd}(s)(k+2)^\frown \dots$ is the sequence of integers played after the last eraser.
- As before, Player $\texttt{II}$ wins if he guesses correctly the image of the sequence of Player $\texttt{I}$, so th winning set is:

$$W = \left\{ s \in T \,\middle|\, f(s^{\texttt{I}}) = s^{\texttt{II}} \right\}$$

This definition corresponds to the tree game by requiring that the tree produced by Player $\texttt{II}$ is finitely branching at the root but all other nodes have at most one child Each branch thus corresponds to to one attempt between two erasers.

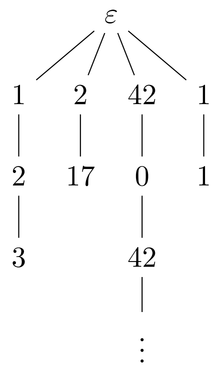


Figure 5: A possible final tree in the eraser game.

Theorem 4.6 (Andretta)**.** Let $f : \omega^\omega \to \omega^\omega$ be a function, then Player $\texttt{II}$ has a winning strategy in the backtrack game $G_{\Uparrow}(f)$ if and only if there exist a $\mathbf{\Pi}^0_1$ partition $(A_n)_{n\in\omega}$ such that $f{\restriction}_{A_n}$ is continuous.

Proof. Let $f : \omega^\omega \to \omega^\omega$ be a function such that Player $\texttt{II}$ has a winning strategy τ in $G_{\Uparrow}(f)$. Let E_n be the set of inputs such that Player $\texttt{II}$ plays exactly n times the eraser while following the strategy τ. We have

$$E_n = \{x \in \omega^\omega \mid \#_{\Uparrow}(x) = n\}$$

where $\#_{\Uparrow}(x)$ is the function that counts the number of $\Uparrow$ in the image of x.

We notice that the E_n are $\mathbf{\Sigma}^0_2$, as we can write E_n in an other way to witness this fact

$$\{x \in \omega^\omega \mid \exists i \; \forall j > i \; \#_{\Uparrow}(x{\restriction}_j) = n\}\,,$$

since the condition $\#_{\Uparrow}(x\restriction_j) = n$ is clearly clopen as we can tell if it holds or not with a finite amount of information about x, namely, the first j digits.

Since in the Baire space, any $\mathbf{\Sigma}^0_2$ can be written as the disjoint union of $\mathbf{\Pi}^0_1$, we have our partition. Moreover, on each of those E_n, Player I doesn't use the eraser after n times, therefore they play the same game as the continuous game and thus the function is continuous on each E_n.

For the other direction, let $f : \omega^\omega \to \omega^\omega$ such that there is a partition of closed sets $(A_n)_{n\in\omega}$ such that $f\restriction_{A_n}$ is continuous. Informally, the strategy for Player II is to start suppose that Player I will produce a $x \in A_1$ and use the strategy for continuous functions on A_1. If at some point, he notices that Player I doesn't play in A_1, he should erase everything and consider that Player I plays in A_2 and the repeat the process. Notice that since A_1 is closed, Player II will always know in a finite amount of time if $x \notin A_1$. Also, Player I cannot alternate between infinitely many A_n because if N is such that $x \in A_N$, with this strategy Player II will use the eraser N times, and then will never think that $x \notin A_N$.

Formally, the strategy is defined as follows. Let τ_n be a winning strategy in the continuous game $G(f\restriction_{A_n})$. Let $a : \omega^{<\omega} \to \omega$ be helper function that returns the index of the A_n that Player II is considering after a play s:

$$a(s) = \min\{n \in \omega \mid [s] \cap A_n \neq \varnothing\}$$

and define the function that gives the first time that Player II considered the set $A_{a(s)}$.

$$b(s) = \min\{n \in w \mid a(s\restriction_m) = a(s)\}.$$

We define the strategy $\tau : \omega^{<\omega} \to \omega^{<\omega} \cup \{\Uparrow\}$ as

$$\tau(s) = \begin{cases} \Uparrow & \text{if } a(s) = b(s) \\ \tau_n(s\restriction_{\mathrm{lh}(s)-b(s)}) & \text{otherwise.} \end{cases}$$

To prove that this strategy is winning, take $x \in \omega^\omega$ and let $N \in \omega$ be the unique integer such that $x \in A_N$. For n large enough we have $a(x\restriction_n) = N$ otherwise we would have a $N' < N$ such that $x\restriction_n \in A_{N'}$ for all n, and therefore $x \in A_{N'}$ as the A_n are closed. Therefore for this n large enough we have that $\tau(s\restriction_{b(n)}) = \Uparrow$ so the sequence produced by II is cleaned and since τ_N is a winning strategy for in $G(f\restriction_{A_N})$, τ_n will produce $f\restriction_{A_n}(x) = f(x)$. Thus, Player II wins. □

Theorem 4.7 (Jaynes, Rogers)**.** A function $f : \omega^\omega \to \omega^\omega$ is $\mathbf{\Lambda}_{2,2}$ if and only if there is a $\mathbf{\Pi}^0_1$ partition $(A_n)_{n\in\omega}$ of ω^ω such that for all $n \in \omega$, $f\restriction_{A_n}$ is continuous.

Corollary 4.8. A function $f : \omega^\omega \to \omega^\omega$ is $\mathbf{\Lambda}_{2,2}$ if and only if Player II has a winning strategy in $G_{\Uparrow}(f)$.

5 Realising the Wadge hierarchy

5.1 Properties of the Wadge hierarchy

In in the next sections, our goal will be to explicitly construct sets that lay at each level of the Wadge hierarchy (below $\mathbf{\Delta}_3^0$).

To that extent, we will ignore the self dual classes and consider only the non-self dual classes, as we will show later that we can obtain them simply with our knowledge of the non-self dual classes.

However, we first need a few definitions and some simple structural results about the Wadge hierarchy, but those results are not necessarily easy to prove, so this is what this section is dedicated to.

The next lemma is at the core of all the following results.

Lemma 5.1 (Wadge's lemma)**.** If A and B are two Borel sets such that $A \not\leq_{\mathrm{W}} B$, then $B^{\mathsf{c}} \leq_{\mathrm{W}} A$.

Proof. This proposition is a direct consequence of the reducibility as winning strategy in the Wadge game $G(A, B)$ and Borel determinacy. Indeed if $A \not\leq_{\mathrm{W}} B$, then Player `II` doesn't have a winning strategy in $G(A, B)$, therefore, Player `I` has a winning strategy (determinacy), that is on every play y of Player `II`, this strategy constructs x such that $y \in B \iff x \notin A$. Hence, if we swap roles so that now Player `I` is in charge of B and Player `II` is in charge of A, Player `II` can copy the previous winning strategy such that he plays in A if and only if Player `I` plays in B^{c}. But this witnesses the fact that Player `II` has a winning strategy in $G(B^{\mathsf{c}}, A)$ and thus $B^{\mathsf{c}} \leq_{\mathrm{W}} A$. □

Corollary 5.2. The Wadge hierarchy on the Borel sets has no antichains of length three.

Proof. Let $X, Y, Z \subset \omega^\omega$ be three Borel sets and suppose the isn't any of them that reduces to an other. So for instance $X \not\leq_{\mathrm{W}} Y$ and $Y \not\leq_{\mathrm{W}} Z$. By using Wadge's lemma, $Y^{\mathsf{c}} \leq_{\mathrm{W}} X$ and $Z^{\mathsf{c}} \leq_{\mathrm{W}} Y$, however, if $Z^{\mathsf{c}} \leq_{\mathrm{W}} Y$, then $Z \leq_{\mathrm{W}} Y^{\mathsf{c}}$, by using the exact same game. Therefore, we have $Z \leq_{\mathrm{W}} Y^{\mathsf{c}} \leq_{\mathrm{W}} X$, contradicting the fact that Z and X are not comparable. □

Theorem 5.3. The Wadge hierarchy hierarchy is well founded, that is, there is no infinite sequence

$$A_0 >_{\mathrm{W}} A_1 >_{\mathrm{W}} A_2 >_{\mathrm{W}} A_3 >_{\mathrm{W}} \ldots$$

Proof. Suppose there exist an infinite descending chain $A >_{\mathrm{W}} A_0 >_{\mathrm{W}} A_1 >_{\mathrm{W}} \ldots$. For all integers i, $A_i^0 = A_i$ and $A_i^1 = A_i^{\mathsf{c}}$, so that we can easily change between a set an its complement. Indeed, if we have $\varepsilon \in \{0, 1\}$, then $A_i^{1-\varepsilon}$ is the complement of A_i^ε. Notice that, by Wadge's lemma, we have

$$\begin{aligned} A_i >_{\mathrm{W}} A_{i+1} &\implies A_i \not\leq_{\mathrm{W}} A_{i+1} \\ &\implies A_{i+1}^{\mathsf{c}} \leq_{\mathrm{W}} A_i \end{aligned}$$

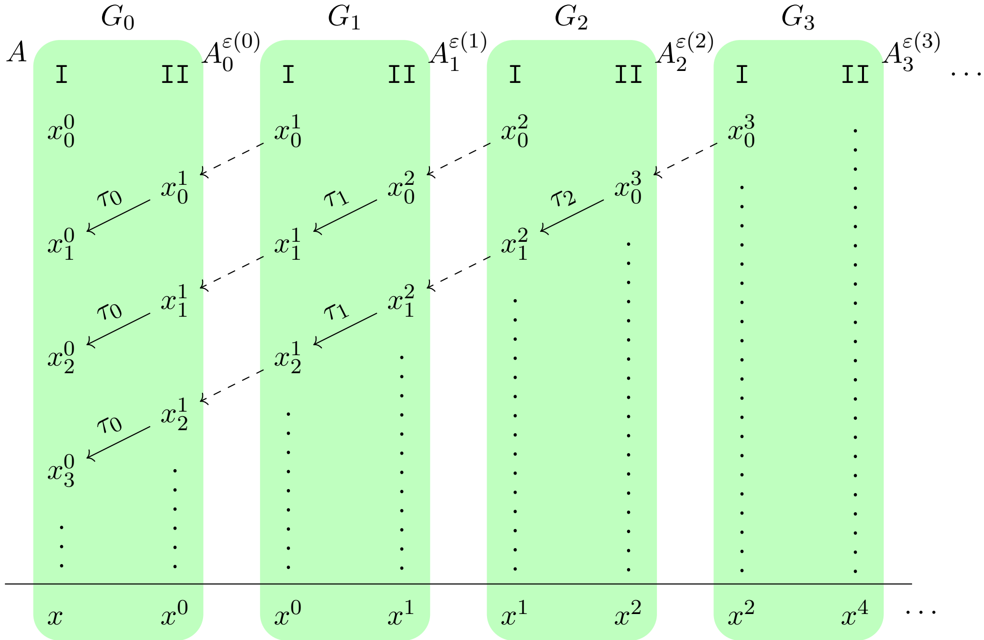


Figure 6: The composition of the sequence of games.

But the last reduction is strict as $A_{i+1}^{\complement} \equiv_{\mathrm{W}} A_i$ would imply that $A_i \equiv_{\mathrm{W}} A_i^{\complement}$ (since our first reduction is the same as $A_i^{\complement} >_{\mathrm{W}} A_{i+1}^{\complement}$) which in turn implies that $A_i^{\complement} \equiv_{\mathrm{W}} A_{i+1}^{\complement}$ which contradicts our first line. Following this, one can see that for any $\varepsilon, \varepsilon' \in \{0, 1\}$, we have $A_i^{\varepsilon} >_{\mathrm{W}} A_{i+1}^{\varepsilon'}$.

Now let $\varepsilon \in 2^{\omega}$ be an infinite sequence of zeros and ones. By the previous points,

$$A >_{\mathrm{W}} A_0^{\varepsilon(0)} >_{\mathrm{W}} A_1^{\varepsilon(1)} >_{\mathrm{W}} A_2^{\varepsilon(2)} \dots$$

Since for every i, $A_i^{\varepsilon(i)} >_{\mathrm{W}} A_{i+1}^{\varepsilon(i+1)}$, it is always Player I that has a winning strategy in $G(A_i^{\varepsilon(i)}, A_{i+1}^{\varepsilon(i+1)})$. Here we have a sequence of games in which Player I always has a winning strategy

$$G(A, A_0^{\varepsilon(0)}) \quad G(A_0^{\varepsilon(0)}, A_1^{\varepsilon(1)}) \quad G(A_1^{\varepsilon(1)}, A_2^{\varepsilon(2)}) \quad G(A_2^{\varepsilon(2)}, A_3^{\varepsilon(3)}) \quad \cdots$$

We will call those games $G_0, G_1, G_2, \dots$ and we will refer to "the game on the right" to talk about G_{i+1} while considering G_i, as we picture them with ω Player I and Player II in a row each playing the game G_i. In each game G_i, let τ_i be the winning strategy of Player I. Notice that in each of those games, the set of which Player II is in charge is always the set of which Player I is in charge in the game on the right. We will construct a play for Player II in the first game, $G_0 = G(A, A_0^{\varepsilon(0)})$ by constructing plays for all the other games. The process is pictured in Figure 6.

In each game G_i, we set that Player II plays exactly the same sequence as the Player I at its right. That is, in the first game Player I plays $x_0^0 = \tau_0(\varnothing)$, according to his winning strategy. To know what Player II answers, he looks at his right and see that this (other) Player I just played $x_0^1 = \tau_1(\varnothing)$, according to his winning strategy, so our Player II just copies this move and also plays x_0^1. At that moment our first Player I know what to answer using his winning strategy and plays $x_1^0 = \tau_0(\langle x_0^1 \rangle)$. Once again our first Player II looks at his right to know what the second Player I played. However, this move depends on what the second Player II played, but the second Player II just copies the first move of the third Player I $x_0^2 = \tau_2(\varnothing)$, thus the second Player I, following his winning strategy, plays $x_1^1 = \tau_1(\langle x_0^2 \rangle)$, which is then copied by our first Player II. To this move the first Player I follows once again his strategy and plays $x_2^0 = \tau_0(x_1^1) = \tau_0(\tau_1(\tau_2(\varnothing)))$.

More generally, we can define the k-th play of Player II in G_i as

$$x_k^i = \begin{cases} \tau_{i+1}(\varnothing) & \text{if } k = 0 \\ \tau_{i+1}\left(x_{k-1}^{i+1}\right) & \text{otherwise} \end{cases}$$

or equivalently as

$$x_k^i = \big(\tau_{i+1} \circ \tau_{i+2} \circ \cdots \circ \tau_{i+k+1}\big)(\varnothing).$$

The second form expresses more clearly that those plays are well defined. At the end of this game, each Player II will have played in G_i the sequence $x^i = (x_0^i, x_1^i, \dots)$, and the Player I will have played exactly the sequence x^{i-1}. If $i = 0$, Player I plays the sequence x with $x_k = \tau_0(x_{k-1}^0)$.

We are now interested in knowing whether $x_0 \in A$ or not, depending on our sequence ε. Notice that, since τ_0 is a winning strategy for Player I in G_0.

$$x \in A \iff x^0 \notin A_0^{\varepsilon(0)}.$$

Similarly, since τ_1 is also a winning strategy in G_1,

$$x \in A \iff x^0 \notin A_0^{\varepsilon(0)} \iff x^1 \in A_1^{\varepsilon(1)}$$

More generally, let $k \in \omega$ then

$$\begin{cases} x \in A \iff x^k \notin A_k^{\varepsilon(k)} & \text{if } k \text{ is even} \\ x \in A \iff x^k \in A_k^{\varepsilon(k)} & \text{if } k \text{ is odd.} \end{cases}$$

Modifying ε. Now suppose that, instead of playing this game with the sequence $\varepsilon \in 2^\omega$, we play it with $\varepsilon' \in 2^\omega$, such that ε and ε' have the same values, except for one digit, say $\varepsilon(N) \neq \varepsilon'(N)$. In this game, all our players play the sequences $x', x'^0, x'^1, \dots$, however, for any $n > N$, we actually have $x'^n = x^n$, since the play in G_n depend only on the plays G_k with $k \geq n$ and the values $\varepsilon(k) = \varepsilon'(k)$. If we look more closely at the game G_{N+1}, we have that Player I plays x'^N and Player II plays $x'^{N+1} = x^{N+1}$. There are two cases, either $x^{N+1} \in A_{N+1}^{\varepsilon(N+1)}$ or $x^{N+1} \notin A_{N+1}^{\varepsilon(N+1)}$:

- If $x^{N+1} \in A_{N+1}^{\varepsilon(N+1)}$, then $x^N \notin A_N^{\varepsilon(N)}$ and $x'^N \notin A_N^{\varepsilon'(N)} = A_N^{1-\varepsilon(N)}$, so $x'^N \in AN$, that is inverting a single digit of ε, transforms $x^N \notin A_N^{\varepsilon(N)}$ into $x'^N \in A_N^{\varepsilon(N)}$.

- If $x^{N+1} \notin A_{N+1}^{\varepsilon(N+1)}$, then we have the exact opposite, that is, we get $x^N \in A_N^{\varepsilon(N)}$ and $x'^N \notin A_N^{\varepsilon(N)}$.

Either way, all the x^n with $n \leq N$ change membership in the $A_n^{\varepsilon(n)}$, that is

$$x^n \in A_n^{\varepsilon(n)} \iff x'^n \in A_n^{\varepsilon(n)}$$

In particular, if we flip any one bit of ε, then $x \in A \iff x' \notin A$.

A Lipschitz function In the last few paragraph, we have constructed a function, $f : 2^\omega \to \omega^\omega$, that maps the sequence $\varepsilon \in 2^\omega$ to the sequence $x \in \omega^\omega$, except that we did not write the dependency on ε everywhere. We claim that this function is in fact a Lipschitz function. Indeed, if we fix $\varepsilon\restriction_n$, the first n digits of ε, then the first n digits of $f(x)$ will not change, as they depend only on the strategies chosen for G_k for $k < n$, which depend on whether the game is played with A_k^0 or A_k^1 and thus on $\varepsilon(k)$, but not on any game after G_n. Therefore, $x\restriction_n$ is also fixed, and thus the function is Lipschitz.

A flip set. Let us consider the set

$$B = \{\varepsilon \in 2^\omega \mid f(\varepsilon) \in A\} = f^{-1}(A).$$

According to what we said before, this set is a **flip set**, that is, a set such that whenever $\varepsilon, \varepsilon' \in 2^\omega$ have exactly one difference, then exact one of $\varepsilon, \varepsilon' \in B$. That is, every time we pick an element in the set, if we flip any of its bits, we have an element outside the set. On the other hand, if we take any element outside of the set and change any of its digits, then we end up in the set.

The function f witness the fact that $B \leq_{\mathrm{W}} A$ as it is continuous, and therefore if A is Borel, then B must also be a Borel set. However, we'll see that a flip set cannot be a Borel set, which will conclude the proof.

A flip set is never a Borel set In order to show that a flip set is never a Borel set, we show that in the Banach-Mazur game of B, the game where each player, in turn play a sequence of integers and Player `II` wins if and only if the concatenation of their sequences is in B, is not determined. To show that neither player have a winning strategy, we will only show that Player `II` doesn't have a winning strategy, as the proof for the other case is very similar.

Assuming that Player `II` has a winning strategy τ in $BM(B)$, we define the strategy σ for Player `I` that will be beat the strategy τ. To that extent, we play two instances of the game $BM(B)$, G and G', where Player `II` follows τ and Player `I` plays as follows.

The process is illustrated in Figure 7.

We denote by $s_k \omega^{<\omega^{<\omega}}$ the moves until the k-th turn of the game G and s'_k the moves until the k-th turn of G'. In G, Player `I` starts by playing $s_0 = \langle 0 \rangle$, to which Player `II` answers with $s_1 = \tau(\langle s_0 \rangle)$. We use those to construct the first move of Player `I` in G', $s'_0 = 1^\frown s_1$, which is exactly $s_0{}^\frown s_1$, but with the first 0 changed to a 1. To the

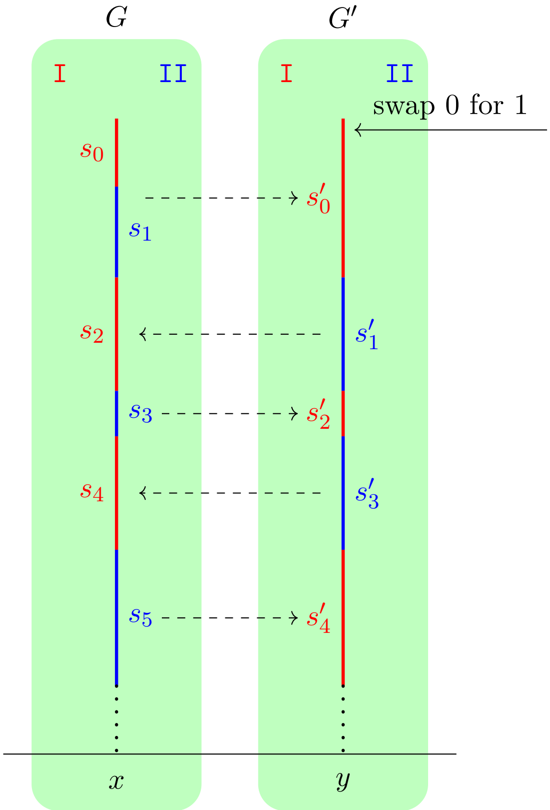


Figure 7: Two instances of the Banach-Mazur game of B produce two sequence with exactly one difference.

move s'_0, Player `II` follows his strategy and plays $s'_1 = \tau(\langle s'_0 \rangle)$. For all the subsequent moves, in G, Player `I` copies the last move from Player `II` in G', that is $s_{2k} = s'_{2k-1}$ to which Player `I` answers with $s_{2k+1} = \tau(s\restriction_{2k+1})$. In the game G', Player `I` always plays $s'_{2k} = s2k + 1$, the last move of Player `II` in the other game, to which `II` answers with his winning strategy.

Since τ is a winning strategy, both sequences $x = s_0{}^\frown s_1{}^\frown \dots$ and $y = s'_0{}^\frown s'_1{}^\frown \dots$ should belong to B. However, by construction we have that $s'_0 = 1^\frown s_1$ we have $s'_{2k} = s_{2k+1}$ and $s'_{2k+1} = s_{2k+2}$. This means that when concatenated,

$$y = 1^\frown s_1{}^\frown s_2{}^\frown s_3{}^\frown s_4{}^\frown \dots$$

and

$$x = 0^\frown s_1{}^\frown s_2{}^\frown s_3{}^\frown s_4{}^\frown \dots$$

Therefore they differ by only one digit and cannot both belong to B, as B is a flip set. $\square$

Definition 5.4. Let $A \subset \omega^\omega$ be a set, $x \in A$ and $n \in \omega$. We define $A_{(x\restriction_n)}$ as the set of sequences in A that starts with $x\restriction_n$.

$$A_{(x\restriction_n)} = \{s \in \omega^\omega \mid s^\frown(x\restriction_n) \in A\}$$

This can be seen as the set of sequences that one player can play after playing $x\restriction_n$ that stays in A.

Proposition 5.5. For any set $A \subset \omega^\omega$ and a sequence $s \in \omega^{<\omega}$,

$$A_{(s)} \leq_{\mathrm{W}} A$$

Proof. Player `II` has a simple winning in $G(A_{(s)}, A)$, has we can first play s and then copy the moves of Player `I`. Thus if Player `I` plays $x \in \omega^\omega$, Player `II` will play $s^\frown x$ and by definition,

$$s^\frown x \in A \iff x \in A_{(s)}.$$

□

This proposition isn't very surprising, as it is clear than once we have played a given sequence s, we are as strong or less strong as before, meaning that we can reduce less sets. However, it is important to notice that some sequences leave the player in charge with strictly less possibilities, i.e. $A_{(s)} <_{\mathrm{W}} A$ whereas some are equivalent to the starting position, that is, $A_{(s)} \equiv_{\mathrm{W}} A$.

Definition 5.6. For a set $A \subset \omega^\omega$, we say that a sequence $x \in \omega^\omega$ is **idempotent** if for all integer $n \in \omega$, we have that $A \equiv_{\mathrm{W}} A_{(x\restriction_n)}$.

Similarly, a finite sequence $s \in \omega^{<\omega}$ is idempotent if $A \equiv_{\mathrm{W}} A_{(s)}$.

An idempotent branch (finite or infinite) is a branch along which a player can play without "loosing strength", that is even after playing this sequence, he can still reduce the same sets as before.

For instance if we consider the open set X of sequences that contains a 1, then the branch that contains only zeros is idempotent as if A is any set, and Player `II` has a winning strategy in $G(A, X)$ then he also has a winning strategy in $G(A, X)$ but where he starts with having played $s = (0, \dots, 0)$ already. This is the same as the game $G(A, X_{(s)})$, but it can be useful to think about s as a prefix in a game, that has already been played.

On the other hand, if X is any $\boldsymbol{\Delta}^0_1$ set, and we take any sequence $x \in \omega^\omega$ we have seen that we know with only a finite prefix whether x belongs to X or not, therefore, for some n we have either $X_{(x\restriction_n)} = \varnothing$ or $X_{(x\restriction_n)} = \omega^\omega$ which are both lower in the Wadge hierarchy, being the two initial degrees.

Definition 5.7. We define the set of idempotent finite sequences of a set $A \subset \omega^\omega$ as

$$\mathrm{Init}_A = \left\{ s \in \omega^{<\omega} \,\middle|\, A_{(s)} \equiv_{\mathrm{W}} A \right\}$$

We call this set Init_A because we see it as the set of all possible where a player wouldn't loose anything it they were written initially in the game. That is, a player has the same 'chance' of winning the game when starting with any sequences of Init_A.

Proposition 5.8. The set Init_A is a tree.

Proof. Let $u \subset s \in \mathrm{Init}_A$. We only need to prove that $u \in \mathrm{Init}_A$. Let t be the sequence such that $s = u^\frown t$, but since

$$A \geq_{\mathrm{W}} A_{(u)} \geq_{\mathrm{W}} \left(A_{(u)}\right)_{(t)} \geq_{\mathrm{W}} A_{(s)} \geq_{\mathrm{W}} A,$$

we must have that $A \equiv_{\mathrm{W}} A_{(u)}$. □

The third important result that we will use multiple times is the following, highlighting the link between self-duality and well-foundedness of the tree of idempotent sequences.

Theorem 5.9. A set $A \subset \omega^\omega$ is self-dual if and only if Init_A is well founded.

Equivalently, A is non-self-dual if and only if it admits an infinite idempotent sequence.

Proof. The first direction has a short proof, but the other direction is very similar to the proof of the previous theorem, however a bit more technical. We give here only the outline of this part of the proof.

Proof of $\Longleftarrow$ Towards a contradiction, assume that there exists A non-self-dual such that Init_A is well founded. We consider the game $G(A, A^{\mathsf{c}})$, where Player `I` should have a winning strategy, since A is non-self-dual. We will however construct a winning strategy for Player `II`. The strategy goes as follows:

- While the sequence s produced by Player `I` belongs to Init_A, that is $A_{(s)} \equiv_{\mathrm{W}} A$, Player `II` passes his turn.
- Once Player `I` plays as sequences $s \notin \mathrm{Init}_A$, he is in charge of $A_{(s)} <_{\mathrm{W}} A$ and therefore, $A_{(s)} <_{\mathrm{W}} A^{\mathsf{c}}$, so Player `II` has a winning strategy τ in $G(A_{(s)}, A^{\mathsf{c}})$, so on all the following moves, Player `II` plays exactly like τ.

This strategy is winning, because after a finite number of turns, Player `I` will play $s \notin \mathrm{Init}_A$, and at this moment, Player `II` still hasn't played anything, so he can indeed copy the strategy τ. If Player `I` plays the sequence $s^\frown x \in \omega^\omega$, the response from Player `II` is $y = \tilde{\tau}(x)$ with

$$y \in A^{\mathsf{c}} \Longleftrightarrow x \in A_{(s)} \Longleftrightarrow s^\frown x \in A,$$

which contradicts that A is non-self-dual.

Proof of $\implies$ The idea of the proof is similar to the last theorem. Assume there exists a set A, self-dual and such that Init_A has an infinite branch, $x \in [\mathrm{Init}_A]$. Because it is self-dual, we have a lot of equivalences

$$A \equiv_{\mathrm{W}} A^{\mathsf{c}} \equiv_{\mathrm{W}} A_{(x\restriction_n)} \equiv_{\mathrm{W}} A^{\mathsf{c}}_{(x\restriction_n)}$$

where $n \in \omega$ is any integer. The idea is to once again have an infinite sequence $A_i^{\varepsilon(i)}$ with some well chosen A_i and $\varepsilon(i)$ indicating whether we take the complement or not. All our A_i will be of the form $A_{(x\restriction_n)}$, so that

$$A \leq_{\mathrm{W}} A_0^{\varepsilon(0)} \leq_{\mathrm{W}} A_1^{\varepsilon(1)} \leq_{\mathrm{W}} \dots$$

In each game $G_i = G(A_{i-1}^{\varepsilon(i-1)}, A_i^{\varepsilon(i)})$, Player `II` has a winning strategy (as opposed to Player `I` last time), so we would like to devise a play for the Player `I` such that we obtain the same contradiction, the set of $\varepsilon \in 2^\omega$, such that in the first game Player `I` plays a sequence $x \in A$ is a flip set. We would again like that the Player `I` copies the player on its right with a winning strategy (Player `II` this time), but for this we need to make sure that the Player `II` on its right doesn't pass, otherwise our Player `I` cannot copy its move.

To that extent, we set $A_0 = A_{(u_0)}$ where u_0 is the sequence such that according to Player `II`'s winning strategy, he plays his first move strait away.

In order to define the other moves, we 'just' set $A_i = A_{(u_i)}$ with a u_i large enough to make all the player on the games on the right play, whatever the values of $\varepsilon(k)$ with $k < i$ are. This is always possible, because there are finitely may possibilities for the $\varepsilon(k)$ and each have a finite sequence such that all the Player `II` play a 'non-pass' move.

We then conclude with the same argument, using the fact that changing a bit from ε, changes the membership of x, and thus $\{\varepsilon \in 2^\omega \mid x \in A\}$ is a flip set, which is impossible for a Borel set. □

Definition 5.10. Le A be a non-self-dual Borel set. We define the **rank** of A, $\mathrm{rk}(A)$ by induction on the ordinals.

- If A belongs to an initial degree of the hierarchy, that is, if $A = \varnothing$ or $A = \omega^\omega$, we set $\mathrm{rk}(A) = 1$.
- Otherwise, we set

$$\mathrm{rk}(A) = \sup \left\{ \mathrm{rk}(B) + 1 \,\middle|\, B <_{\mathrm{W}} A \right\}$$

This definition is sound because of the well-foundedness of the Wadge hierarchy. If the Wadge hierarchy were ill-founded, it would be impossible to inductively define the rank of a set in an infinite descending chain, as it it would require that we already know the ranks of all the sets bellow, and this, for each set in the chain.

We have thus proven that the Wadge hierarchy is very similar as the Borel hierarchy in the sense that it is a well founded order with antichains of length at most two at every rank, and it goes like this for a given initial segment of the ordinals. We haven't proven how high it goes, and not even that it doesn't stop after the first few levels, but we will see that it doesn't stop after the first few levels and that it is much finer than the Borel hierarchy.

5.2 Automatic sets

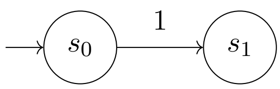
In this section, we develop a new tool to describe and construct new sets. We call them automatic sets, because of the their great similarity with automata. They are, however, different from automata in two ways.

First, they are infinite objects, with at most countably many states, as opposed to the very finite nature of automata. They will also read infinite words, similarly to ω-automata like a Muller, Büchi or parity automaton.

Second, their interest is not of expressing certain kind of languages, unlike the finite state machines, which recognise exactly regular languages, our "automata" will be able to recognise any languages, quite easily. Their goal is indeed to be very expressive so that we can construct new sets (or languages) in a simple, unified manner.

We will introduce gently to automatic sets with many examples. For the readers who swish to read directly the formal definition, see Definition 5.13, page 62.

The basis of automatic sets Similarly to finite state machines, an automatic set consist of a set of states, however it does not need to be finite as we will consider countable sets of states. Each automatic set's purpose is to read a word, $x \in \omega^\omega$ and update its state accordingly, using defined deterministic transitions. That is, a simple automatic set with two states s_0 and s_1 could be



Where the initial state is s_0 and on any input $x \in \omega^\omega$, if the current state of the automatic set is s_0 and $x_n = 1$, it move to the state s_1, otherwise it does not change state as there is no labelled arrow for other integers, so the next state is still s_0. Once it is in s_1, it stays there for the rest of the input.

We often think of automatic sets as countable directed graphs, along which one moves by inputing one of the transition label.

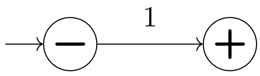
Acceptance and language An important notion is the notion of accepting state. Similarly to finite state machines, a state can be accepting or rejecting. An straightforward way to define whether an automatic set accepts an input x or not would be to say that, if after a finite amount of time, it stays forever in an accepting state, then x is accepted. Similarly, if it stays forever in a rejecting state, x should surely not be accepted.

However, this doesn't take into account an important case, when the states keep alternating. We will need to specify what happens in this case, as saying that it always rejects is an option, but not the most expressive one and it would quickly limit us.

For a given automatic set A, we define the language of A, as

$$\mathcal{L}(A) = \{x \in \omega^\omega \mid A \text{ accepts } x\}$$

With this in mind and the convention that we represent the accepting states of an automatic set with a plus sign and the rejecting states with a minus sign, the previous automatic set would be

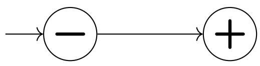


if we set that s_1 is the only accepting state. The language of this automatic set is then the set of sequences that contain a one, which is an open set.

Not specifying the transitions We notice that the fact that the previous automatic set describes an open set doesn't depend of what is written on the transition, as long as it is not the empty set or the set of integers. Indeed, in this case the recognised language would be $\varnothing$ or ω^ω respectively and those two sets from the initial degrees of the Wadge hierarchy, strictly below the open sets. On the other hand, for any set $T \in \mathcal{P}(\omega) \setminus \{\varnothing, \omega^\omega\}$, the automatic set describes the set of sequences that contain at least one of the integers of T, which is always an open set.

It is true in greater generality, that changing the labels doesn't change the Wadge degree of the language, and we will thus consider that each transition is labeled with exactly one integer, that correspond to the state that it leads to. We also want that two transitions that start at the same state have different integers so that it stays deterministic, and that there are always a way to stay in the same state.

For instance, an automatic set for the open sets is:



Indeed if this automatic set accepts $x \in \omega$, it means that after finitely many inputs, say $x{\restriction_n}$ the automatic set switched from the rejecting state to the accepting state. This means that any sequence that extends $x{\restriction_n}$ ends up in the accepting state, as there is no way to leave it. Therefore, $[x{\restriction_n}]$ is contained in the language and thus the language is an open set. It is also not any lower level as it is not closed: we need to wait infinitely many turns to be sure that an input is not accepted.

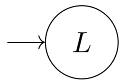
However, which open set it represents precisely is not well defined. In order to get one well defined open set from this automatic set, we need to specify the transition rules, and once they are specified, we can talk of the language of the automatic set, which is then a well defined set. In practice, we will not specify those transitions before using $\mathcal{L}(\mathcal{A})$, and it should be understood that one can choose any set of transition rules so that the argument applies.

Arbitrary sets as states There are two more ways we make our automatic sets more powerful. The first is to allow for arbitrary sets as states, and not only accepting and rejecting states. The main idea is that, while we are inside a state, the input that is read is not used, except if it contains a transition to an other state. Instead, if at some point the state of the automatic set does not change, we will use the input while in this state to determine whether or not we accept a sequence.

For instance, if an automatic set $\mathcal{A}$ stays in the state $A \subset \omega^\omega$ at some point on an input $x = s^\frown x' \in \omega^\omega$, where x' is the part of the input spent in the state A, we set that

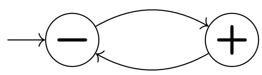
$$\mathcal{A} \text{ accepts } x \iff x' \in A.$$

This let us construct automatic set of arbitrary complexity, and explains why it is not interesting to look at automatic sets as automata that recognise a particular class of languages, as for any language $L \subset \omega^\omega$, it is simple to construct an automatic set that recognises precisely L:



Note that with the generalisation, our rejecting state $\ominus$ and accepting state $\oplus$ from before correspond respectively to using $\varnothing$ or ω^ω as a state, as it is impossible for an input to end up in $\varnothing$, and likewise, any input will belong to ω^ω.

Always alternating automatic sets One may notice that everything works well in the previous examples, because we mostly used finite automatons, there was only finitely many states, with the flow of the automatic set going only one way. No cycles, no loops, but that may not always be the case. In those cases, one should be careful to correctly define the accepting state of an automatic set. If we take for instance

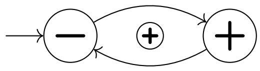


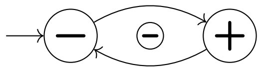
On an input $x \in \omega^\omega$, there are three cases, either at some point it stays in the accepting state until the end, or it stays in the rejecting state until the end, or it alternates infinitely many times between the two.

Here, there are essentially two interesting ways of choosing whether x is accepted or not:

- we could say that x is accepted if and only if the automatic set stays in the accepting state at some point. So the other two cases (infinitely alternating and keep rejecting) are rejecting.
- Or we could say that if the automatic set alternates between the two states infinitely many times, it also accepting. This way, only staying in the rejecting state doesn't accept x.

Both of those definitions are interesting and there isn't one that should be preferred over the other, so we only need to precise what do we do when the state alternates infinitely many times. This gives us two automatic sets,





where we denote by a plus or minus between the arrow, what happens when alternating between the two states infinitely. We will see in Proposition 5.12 that those two automatic set correspond respectively to a $\boldsymbol{\Sigma}^0_2$ and a $\boldsymbol{\Pi}^0_2$ set.

More generally, when the automatic set can alternates between states, we use an acceptance condition $\mathrm{Acc} \subset \mathcal{P}(S)$ where S is the set of states of the automatic set. If an input alternates infinitely many times between states it is accepted exactly when the sets of states between which it alternates belongs to Acc. For instance, the two previous automatic sets we would have respectively $\mathrm{Acc} = \{⊖, ⊕\}$ and $\mathrm{Acc} = \varnothing$.

This acceptance condition is the same as for Muller automata, which makes the our automatic sets very similar to Muller automata, except for the fact that they can have arbitrary languages as states and have countably many states.

Wadge games with automatic sets Now that we have a correspondence between automatic sets and sets, through their language, we can include then in our favorite tool: games. This will be easy however, because automatic sets are mostly an "automaton-like" way of describing sets, so we extend the definition of the Wadge game ton automatic set $\mathcal{A}$, by saying that we play exactly the same way as we would with $\mathcal{L}(\mathcal{A})$.

In a game between a set and an automaton,

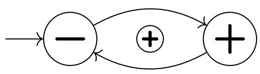
Definition 5.11. Let $\mathcal{A}$ be any automatic set and $X \subset \omega^\omega$ be a set. We define the Wadge game between X and $\mathcal{A}$ as the game $G(X, \mathcal{A})$ where Player I is in charge of an usual set and Player II is in charge of the automatic set $\mathcal{A}$.

In this game, a play of Player II is a sequence of states $s = (s_0, s_1, \dots)$ of the automatic set A, such that there is always a transition in the automatic set between s_n and s_{n+1}.

At the end, let $x \in \omega^\omega$ be the sequence played by Player I. We say that Player II wins if

$$x \in B \iff A \text{ is accepting } s$$

Proposition 5.12. The language recognised by the automatic set



is a $\boldsymbol{\Sigma}^0_2$-complete set.

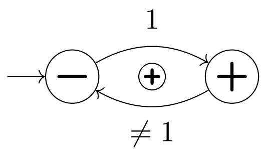
Proof. Let $\mathcal{A}$ be this automatic set and let $X = \{x \in \omega^\omega \mid \exists^\infty n \; x(n) = 1\}$ be the set of sequences with infinitely many ones. We have already proven that X is $\mathbf{\Sigma}^0_2$-complete so we only need to show that Player `II` has a winning strategy in both $G(\mathcal{A}, X)$ and $G(X, \mathcal{A})$.

For $G(\mathcal{A}, X)$, the strategy for Player `II` can be to play a one every time Player `I` visits the accepting state, and to play a zero every times Player `I` visits the rejecting state. This way, Player `II` plays infinitely many ones exactly when Player `I` visits infinitely many times the accepting state, which corresponds to the accepting condition.

For $G(X, \mathcal{A})$, we do almost the same thing, copying what Player `I` does. When Player `I` plays a one, we go to the accepting state, and when he doesn't, we go to the rejecting state. This is also clearly winning for Player `II`. □

This proofs shows two interesting things. First, we can see the importance of accepting and rejecting states as a guess of whether we end up in the set X or not. That is every time the other player plays a one, we should lean towards the idea that he may plays infinitely many more and thus accepting. On the other hand, no one can argue for long that a sequence ends up in X, if the other player keeps plays some 42s, otherwise they loose the game. The great strength of those automatic sets is to consider only whether or not a sequence is likely to end up in a set, without considering too many details.

The second thing to note is that it is not surprising that $\mathcal{A}$ reduces this simply to X (and X to $\mathcal{A}$), as there is a labeling of the transitions of $\mathcal{A}$ that gives precisely X, namely,



where the $\neq 1$ means everything but one. We therefore see this automatic set as a $\mathbf{\Sigma}^0_2$, but if we want to obtain a precise $\mathbf{\Sigma}^0_2$, we only need to give a labeling of the edges.

Through this point of view, we can see the games with automatic sets as a Wadge game where the player is allowed to switch between sets, using some transition tokens. Those can be made to fit into the original definition of the Wadge game by using a proper encoding of those token and noticing that there are only countably many transitions, so for instance, tokens could be encoded as even numbers and the play of a digit as an odd number.

Formal definition We now define formally automatic sets using the intuition gathered from the previous paragraphs. We start we a more general definition, allowing for arbitrary alphabets and not solely ω, and later precise the definition of automatic sets into ω-automatic sets, which will be what we want to use on the Baire space.

Definition 5.13. An **automatic set** $\mathcal{A}$ on an countable alphabet Σ is composed of

- A countable set of **states**, S with a designated **initial state** $s_0 \in S$.
- A coloring $\mathcal{S} : S \to \mathcal{P}(\Sigma^\omega)$ that map each state to a set of sequences on Σ.
- For each state $s \in S$, a function $\mathcal{T}_s : \Sigma \to S$, that associates to each input the 'next' state. The map $\mathcal{T} : s \mapsto \mathcal{T}_s$ is called the **transition function**.
- An **acceptance condition** $\mathrm{Acc} \subset \mathcal{P}(S)$.

We will usually forget the function $\mathcal{S}$ and directly consider that a state is a set of sequences on Σ. We cannot define it as such because two states can be associated to the same set, so in practice we should only make sure to be clear about which state we consider if there are multiple states that are the same set.

We now define a run of an automatic set, on a given input $x \in \Sigma^\omega$.

Definition 5.14. Let $\mathcal{A}$ be an automatic set and $x \in \Sigma^\omega$ be a sequence on Σ. We call x an **input** of $\mathcal{A}$. The **run** of $\mathcal{A}$ on x is a sequence of states $s = (s_0, s_1, \dots) \in S^\omega$ such that s_0 is the initial state of $\mathcal{A}$ and for each $n \in \omega$,

$$s_{n+1} = \mathcal{T}_{s_n}(x_n).$$

That is, s_{n+1} is the next state as given by the transition rule.

Given a run $s \in S^\omega$, we write $\mathrm{Inf}(s)$ for the set of states visited infinitely many times:

$$\mathrm{Inf}(s) = \left\{x \in S \,\middle|\, \exists^\infty n \in \omega \ s_n = x\right\}$$

Definition 5.15. Let $\mathcal{A}$ be an automatic set, $x \in \Sigma^\omega$ be an input, and $s \in S^\omega$ be the run of $\mathcal{A}$ on x. Let

$$N = \min\{n \in \omega \,|\, \forall m \geq n \ x_n = x_m\}$$

be the least integer starting from which the run is constant.

- If such N exists, let x' be the sequence such that $x = (x\restriction_N)^\frown x'$. We say that $\mathcal{A}$ **accepts** x if $x' \in \mathcal{S}(s_N)$, that is, the input after the last change of state is in the set corresponding to that last state.

- If such N does not exist, let $\mathrm{Acc} \subset \mathcal{P}(S)$ be the acceptance condition of $\mathcal{A}$. We say that $\mathcal{A}$ **accepts** x if $\mathrm{Inf}(s) \in \mathrm{Acc}$.

If $\mathcal{A}$ does not accept x, we say that $\mathcal{A}$ **rejects** x.

In some sense, defining whether or not $\mathcal{A}$ accepts an input x is easy in the case that the states stop alternating. If they do, we can simply look at the part of the input that we read while we were in the last state, and check if it is in the set of the state or not.

Then, the decision function helps deciding what to do when it alternated infinitely often. Therefore we do not need to provide a decision function when an automatic set cannot alternate infinitely many times. We also do not need to specify the value of the decision function on runs that are constant at the limit, as we will never use this value.

Definition 5.16. Let $\mathcal{A}$ be an automatic set on the alphabet Σ. The **language** of $\mathcal{A}$ is the set

$$\mathcal{L}(\mathcal{A}) = \{x \in \Sigma^\omega \mid \mathcal{A} \text{ accepts } x\}.$$

Automatic sets on the Baire space We will mostly be interested by automatic sets that work on the Baire space, which will be of a special form.

Definition 5.17. An ω**-automatic set** is an automatic set such that

- The alphabet is $\Sigma = \omega$
- For each state $s \in S$, let T_s be the set of states that are reachable from s, and $f_s : \omega \cup T_s \to \omega$ be any bijection that encodes integers and transitions into integers.
- The sets of each states code sequences of integers, so

$$\mathcal{S}(s) \subset \big(f_s(\omega)\big)^\omega$$

- The transition function is defined as

$$\mathcal{T}_s(n) = \begin{cases} s & \text{if } n \in f_s(\omega) \\ f_s^{-1}(n) & \text{if } n \in f_s(T_s). \end{cases}$$

Such automatic set can be seen as a countable directed graph, where a player can either play sequences of integers $x \in \omega^\omega$ that will be encoded through f_s but also choose to change the state to a neighbouring state. However, we fill never explicitly define explicitly the f_s, and instead consider that the input is made of integers and ‘tokens’ to change between states, which would be bijectively encoded as a sequence of the Baire space.

Now we define an automatic set $\mathcal{A}^{\mathsf{c}}$ that recognises the complement of the language of $\mathcal{A}$, so that $\mathcal{L}(\mathcal{A}^{\mathsf{c}}) = \mathcal{L}(\mathcal{A})^{\mathsf{c}}$.

Definition 5.18. Let $\mathcal{A}$ be an ω-automatic set. We define $\mathcal{A}^{\mathsf{c}}$ to be the same automatic set as $\mathcal{A}$, except that:

- For each state $s \in S$, its corresponding set $\mathcal{S}'(s)$ is
$$\mathcal{S}'(s) = \big(f_s(\omega)\big)^{\omega} \setminus \mathcal{S}(s).$$
- Its acceptance condition is Acc',
$$\mathrm{Acc}' = \mathcal{P}(S) \setminus \mathrm{Acc}$$

Proposition 5.19. Let $\mathcal{A}$ be an ω-automatic set, then
$$\mathcal{L}(\mathcal{A}^{\mathsf{c}}) = \mathcal{L}(\mathcal{A})^{\mathsf{c}}.$$

Proof. Let $x \in \omega^{\omega}$ be an input for $\mathcal{A}$. We need to show that $\mathcal{A}^{\mathsf{c}}$ accepts x if and only if $\mathcal{A}$ rejects x. Let s be the run of x in $\mathcal{A}$, which is the same as the run of x in $\mathcal{A}^{\mathsf{c}}$. Let also $\mathcal{S}$ be the sets of states of $\mathcal{A}$ and $\mathcal{S}'$ be the sets from $\mathcal{A}^{\mathsf{c}}$.

If s is constant at the limit, let s_ω be the limit and y be the part of the input x played after the last change of state. Since there was no change of state while y was played, we have that $y \in \big(f_s(\omega)\big)^{\omega}$, so that by definition,
$$y \in \mathcal{S}(s) \iff y \notin \mathcal{S}'(s).$$

If s is not constant at the limit, let Acc be the decision function of $\mathcal{A}$ and Acc' of $\mathcal{A}^{\mathsf{c}}$. We directly have that
$$\mathrm{Inf}(s) \in \mathrm{Acc}' = \mathcal{P}(S) \setminus \mathrm{Acc} \iff \mathrm{Inf}(s) \notin \mathrm{Acc},$$
so that in both cases, x is accepted if and only if the other automatic set rejects it. □

Games with automatic sets We now extend the definition of the Wadge game to allow for automatic sets to participate.

Definition 5.20. Let $\mathcal{A}$ be an ω-automatic set and X a Borel set. The Wadge game $G(X, \mathcal{A})$ is the game $G(X, \mathcal{L}(\mathcal{A}))$, and similarly if X is also an automatic set.

We consider that

- Moves for Player `I` are integers so that he produces $s^{\mathtt{I}} \in \omega^{\omega}$.
- Moves for Player `II` are inputs for the automatic set $\mathcal{A}$, so Player `II` plays sequences of integers $s_n \in \omega^{<\omega}$ and produces $s^{\mathtt{II}} = s_0{}^\frown s_1{}^\frown s_2{}^\frown \ldots$.

- Player $\mathtt{II}$ wins if $s^{\mathtt{II}}$ is infinite and

$$s^{\mathtt{I}} \in X \iff \mathcal{A} \text{ accepts } s^{\mathtt{II}}.$$

The main interest of this definition, and of the whole shift to automatic set is that we replace the typical condition $x \in A$ for the condition $\mathcal{A}$ accepts x, which is closer of how we think about strategies. It also offers greater comparability of sets since it is simple to create new automatic sets from given sets (or automatic sets) and thus obtain a wide variety of sets with desired properties.

Using sub-trees of the Baire space. A generalisation of ω-automatic sets that we will use once during the proof of Theorem 5.24 is to allow that at a node of the automatic set we we do not have a subset of the Baire space but instead, a subset of a closed set of the baire space. This correspond to changing the tree in which we are playing: instead of using the whole tree of the Baire space, we allow for only a subtree. In practice, if $[T] \subset \omega^\omega$ is a tree, and $X \subset [T]$ is any se we allow states to be labeled $(X, [T])$, meaning that we play inside the tree T and a sequence $x \in [T]$ is accepted if and only if $x \in X$.

This still fits in the frame of our previous definition of ω-automatic sets as we can encode any countably branching tree in the tree of the Baire space, that is, if $T \subset \omega^{<\omega}$ is a non-empty pruned tree there exists a length preserving, monotone surjection $f : \omega^\omega \to T$. With such f, using $f^{-1}(X)$ for a state or $(X, [T])$ is exactly the same, except that maybe some sequences $x \in \omega^\omega$ code for the same sequence $f(x) \in [T]$. Since f is surjective, we always have $x \in f^{-1}(X) \iff f(x) \in X$ thus both versions are equivalent.

5.3 Construction of $\boldsymbol{\Delta}^0_2$ sets

Our next goal is to explicitly construct non-self-dual sets of each countable rank. In order to create sets of a given rank, we will define two operations between sets, $\oplus$ and $\sup^\omega$. Our goal is to have, for A and B two non-self-dual Borel sets

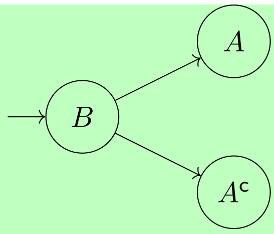
$$\mathrm{rk}(A \oplus B) = \mathrm{rk}(A) + \mathrm{rk}(B)$$

and for $(A_n)_{n\in\omega}$ a countable sequence of non-self-dual Borel sets,

$$\mathrm{rk}(\sup^\omega(A_n)) = \sup\{\mathrm{rk}(A_n) \mid n \in \omega\}$$

We will later reuse those tools to constructs sets of each rank $\alpha < \omega_1^{\omega_1}$.

Definition 5.21. Let A and B be two Borel sets. Let $\mathcal{A}$ be the ω-automatic set defined as



The addition $A \oplus B$ is then

$$A \oplus B = \mathcal{L}(\mathcal{A}).$$

Here we consider $\mathcal{L}(\mathcal{A})$ as the subset of the ω^ω space through any bijection $f : \Sigma \to \omega$, for instance

$$f(x) = \begin{cases} 0 & \text{if } x = A \\ 1 & \text{if } x = A^{\mathsf{c}} \\ x+2 & \text{if } x \in \omega. \end{cases}$$

An other way to define this set without automatic sets would be to explicitly write what it means to be accepted by this automatic set. A input to $\mathcal{A}$ is valid if it contains at most one transition A or A^{c} as each run can change at most once of state. Now one can see that a valid input x is accepted if one of those tree conditions holds:

- $x \in B$.
- $x = s^\frown \mathtt{a}^\frown y$ with $s \in \omega^{<\omega}$ and $y \in A$. Here we denoted by $\mathtt{a}$ the transition to the state A.
- $x = s^\frown \mathtt{a}^{\mathsf{c}\frown} y$ with $s \in \omega^{<\omega}$, $y \in A^{\mathsf{c}}$ and $\mathtt{a}^{\mathsf{c}}$ denotes the transition to the state A^{c}.

Therefore if we use the convention that the addition on a sequence add to each of its digit, that is, $x + 2 = (x_0 + 2, x_1 + 2, \dots)$, we have

$$A \oplus B = \left(\begin{array}{c} \left\{ x+2 \,\middle|\, x \in B \right\} \\ \cup \\ \left\{ (s+2)^\frown 0^\frown (x+2) \,\middle|\, s \in \omega^{<\omega} \wedge x \in A \right\} \\ \cup \\ \left\{ (s+2)^\frown 1^\frown (x+2) \,\middle|\, s \in \omega^{<\omega} \wedge x \in A^{\mathsf{c}} \right\} \end{array} \right)$$

This new "addition" of sets is very natural in the context of a game, as it corresponds to a player starting to play in B and at any point he can change his mind, play the $\mathtt{a}$ token and now play in A or it can play the token $\mathtt{a}^{\mathsf{c}}$ and play in A^{c}. It basically allows the player to change the set that he is in charge in the middle of a game if he wishes.

Lemma 5.22. Let A and B be two Borel sets.

$$(A \oplus B)^{\mathsf{c}} \equiv_{\mathrm{W}} A \oplus B^{\mathsf{c}}$$

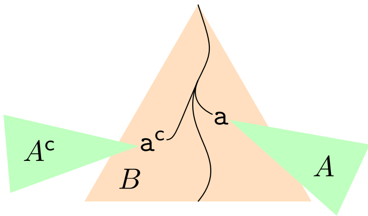
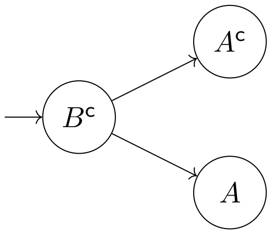


Figure 8: Representation of the tree of $A \oplus B$.

Proof. Let $\mathcal{A}$ be the automatic set from the definition of $A \oplus B$. We have that $(A \oplus B)^{\mathsf{c}} = \mathcal{L}(\mathcal{A})^{\mathsf{c}} \equiv_{\mathrm{w}} \mathcal{L}(\mathcal{A}^{\mathsf{c}})$, but the complement of $\mathcal{A}$ is the automatic set



Which is exactly the automatic set of $A^{\mathsf{c}} \oplus B^{\mathsf{c}}$, but also the automatic set of $A \oplus B^{\mathsf{c}}$, if we swap the position of the two nodes A and A^{c}. □

Lemma 5.23. Let $A, B \subset \omega^\omega$ such that B is non-self-dual, then $A \oplus B$ is also non-self-dual.

Proof. By Theorem 5.9, it suffices to show that $A \oplus B$ has an infinite idempotent sequence. Let x be an infinite idempotent sequence for B. Then a play along this sequence without changing state is also an idempotent sequence for $A \oplus B$, as at any point we can still change state for A or A^{c} and are still in charge of a set equipotent to B in the state B. □

Theorem 5.24. Let A and B be two Borel sets.

$$\operatorname{rk}(A \oplus B) = \operatorname{rk}(A) + \operatorname{rk}(B)$$

Proof. We prove the theorem by induction on the rank of B.

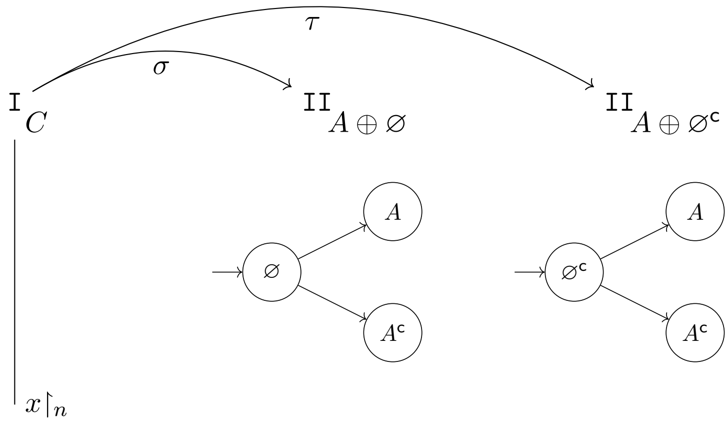
The base case. Let $B = \varnothing$ so that $\operatorname{rk}(B) = 1$. By definition of the rank, we have that

$$\operatorname{rk}(A \oplus \varnothing) = \sup \left\{ \operatorname{rk}(C) + 1 \,\middle|\, C <_{\mathrm{w}} A \oplus \varnothing \right\}$$

Therefore we only need to show that for a non-self-dual set C, $C <_{\mathrm{w}} A \oplus \varnothing$ if and only if $C \leq_{\mathrm{w}} A$. By Wadge's lemma, $C <_{\mathrm{w}} A \oplus \varnothing$ implies both

$$C \leq_{\mathrm{w}} A \oplus \varnothing \qquad C \leq_{\mathrm{w}} (A \oplus \varnothing)^{\mathsf{c}}$$

By Lemma 5.22 we can simplify the rightmost set, $(A \oplus \varnothing)^{\mathsf{c}} \equiv_{\mathrm{W}} A \oplus \varnothing^{\mathsf{c}}$ Because of those two reductions, Player `II` has a winning strategy in the games $G(C, A \oplus \varnothing)$ and $G(C, A \oplus \varnothing^{\mathsf{c}})$. We name them σ and τ respectively. Let x be an infinite idempotent sequence of C, such that $C_{(x\restriction_n)} \equiv_{\mathrm{W}} C$.



Since both σ and τ are winning strategies, it must be the case that, for some $n \in \omega$, either σ leaves the state $\varnothing$ or τ leave the state $\varnothing^{\mathsf{c}}$, otherwise the two automatic sets cannot agree on whether $X \in C$ or not.

Thus, one of the two automatic sets leave its initial states for either A or A^{c}, but since the strategies are winning, it means that either

$$C_{(x\restriction_n)} \leq_{\mathrm{W}} A \quad \text{or} \quad C_{(x\restriction_n)} \leq_{\mathrm{W}} A^{\mathsf{c}}.$$

In either case, since $C \equiv_{\mathrm{W}} C_{(x\restriction_n)}$, we have that $C \leq_{\mathrm{W}} A$ or $C \leq_{\mathrm{W}} A^{\mathsf{c}}$ and thus $\mathrm{rk}(C) \leq \mathrm{rk}(A)$. Thus,

$$\begin{aligned}\mathrm{rk}(A \oplus \varnothing) &= \sup \left\{ \mathrm{rk}(C) + 1 \,\middle|\, C <_{\mathrm{W}} A \oplus \varnothing \right\} \\ &= \sup \left\{ \alpha + 1 \,\middle|\, \alpha \leq \mathrm{rk}(A) \right\} \\ &= \mathrm{rk}(A) + 1.\end{aligned}$$

The recursive case. Suppose now that $\mathrm{rk}(B) \neq 1$, so that $B \neq \varnothing$ and $B \neq \omega^\omega$. By definition of the rank, we have that

$$\mathrm{rk}(A \oplus B) = \sup \left\{ \mathrm{rk}(C) + 1 \,\middle|\, C <_{\mathrm{W}} A \oplus B \right\}.$$

First we show that $\mathrm{rk}(A \oplus B) \geq \mathrm{rk}(A) + \mathrm{rk}(B)$, but it is simple as for any $D <_{\mathrm{W}} B$, we have that $A \oplus D \leq_{\mathrm{W}} A \oplus B$ and thus

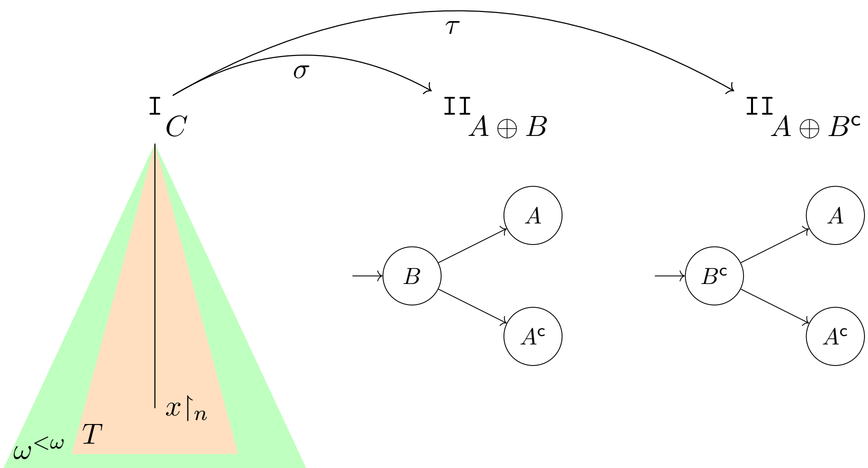
$$\begin{aligned}\mathrm{rk}(A \oplus B) &\geq \sup \left\{ \mathrm{rk}(A \oplus D) + 1 \,\middle|\, D <_{\mathrm{W}} B \right\} \\ &\geq \sup \left\{ \mathrm{rk}(A) + \mathrm{rk}(D) + 1 \,\middle|\, D <_{\mathrm{W}} B \right\} \\ &\geq \sup \left\{ \mathrm{rk}(A) + d + 1) \,\middle|\, d < \mathrm{rk}(B) \right\} \\ &\geq \mathrm{rk}(A) + \mathrm{rk}(B)\end{aligned}$$

Where we have used the induction hypothesis on the second line and the definition of ordinal addition on the last.

For the reverse inequality, our goal is to show that for each non-self-dual set $C <_{\mathrm{w}} A \oplus B$, there is a non-self-dual set $D <_{\mathrm{w}} B$ such that $C \leq_{\mathrm{w}} A \oplus D$. We would therefore have $C \leq_{\mathrm{w}} A \oplus D <_{\mathrm{w}} A \oplus B$, which implies that $\mathrm{rk}(A \oplus B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$.

Let $C <_{\mathrm{w}} A \oplus B$ be a non-self-dual set. Similarly to the base case, we must have both $C \leq_{\mathrm{w}} A \oplus B$ and $C \leq_{\mathrm{w}} A \oplus B^{\mathsf{c}}$, so let σ and τ be the winning strategies in the games $G(C, A \oplus B)$ and $G(C, A \oplus B^{\mathsf{c}})$ respectively.

We have the following picture:



Let T be the set of finite plays $s \in \omega^{<\omega}$ of Player I such that σ doesn't leave the state B and τ doesn't leave B^{c}.

If $[T]$ is empty, then $C \leq_{\mathrm{w}} A \oplus \varnothing$ as a strategy consists in waiting that σ leave B, which always happen, and then move to A or A^{c} accordingly, copying the moves from σ.

If $[T]$ is non empty, we will show that $C \leq_{\mathrm{w}} A \oplus (C \cap [T], [T])$, where $(C \cap [T], [T])$ being the set $C \cap [T]$ but as a subset of $[T]$, and not the Baire space. There are a few things to show.

1. First, we need to prove that $(C \cap [T], [T]) <_{\mathrm{w}} B$.

2. Then we need to show that $C \leq_{\mathrm{w}} A \oplus (C \cap [T], [T])$, which will be simpler than the first point.

3. Finally prove that $(C \cap [T], [T])$ is non-self-dual, otherwise, we can't use its rank.

Proof of 1. In order to show that $(C \cap [T], [T]) <_{\mathrm{w}} B$, we show that $(C \cap [T], [T]) \leq_{\mathrm{w}} B$ and $(C \cap [T], [T]) \leq_{\mathrm{w}} B^{\mathsf{c}}$ by exhibiting a strategy in each game. In fact, we already have those strategies, as σ restricted to $(C \cap [T], [T])$ is a winning strategy in $G((C \cap [T], [T]), B)$ since it doesn't leave B and is winning. Similarly, τ is a winning strategy in $G((C \cap [T], [T]), B^{\mathsf{c}})$.

Proof of 2. To show that $C \leq_{\mathrm{w}} A \oplus (C \cap [T], [T])$, we construct a strategy ρ for Player II in the game $G(C, A \oplus (C \cap [T], [T]))$ by combining σ and τ. The main idea is check whether or not Player I stays in T or not. Let $s \in \omega^{<\omega}$ be a play of Player I. We define our the strategy ρ as

- if $s \in T$, $\rho(s) = s$, we follow the strategy σ.
- if $s \notin T$, let $u \subset s$ be the first prefix of s that does not belong to T. From the definition of T, we have that $\sigma(u)$ or $\tau(u)$ correspond to a transition to the state A or A^{c}. We follow the strategy that transitioned, that is

$$\rho(s) = \begin{cases} \sigma(s) & \text{if } \sigma(u) \notin (C \cap [T], [T]) \\ \tau(s) & \text{otherwise.} \end{cases}$$

This is a winning strategy. Indeed, let $x \in \omega^\omega$ be a sequence produced by Player I. If $x \in [T]$, Player II produces $y = \tilde{\sigma}(x) = \tau(x{\restriction}_0)^\frown\tau(x{\restriction}_1)^\frown\ldots$ but $\tilde{\sigma}$ is the identity on $[T]$, so Player II wins.

If $x \notin [T]$, let $x = u^\frown x'$ with u the least prefix of x such that $u \notin T$. By construction $\rho(x) = t^\frown\mathtt{t}^\frown y$, where $t \in T$, $\mathtt{t}$ is a transition from the state $(C \cap [T], [T])$ to either A or A^{c}, and y is the play inside the state A (or A^{c}). Starting at $\mathtt{t}$, ρ follows exactly σ or τ, and since the start of the sequence doesn't matter as it is forgotten in the acceptation of the automatic set, we have that $x \in C \iff y \in A$ (or $\iff y \in A^{\mathsf{c}}$, depending on the case). In either case, Player II wins.

Proof of 3. Finally, we need to show that $(C \cap [T], [T])$ is non-self-dual. To that extent, consider an idempotent sequence $u \in \omega^{<\omega}$, so that $C_{(u)} \equiv_{\mathrm{w}} C$.

If $u \notin T$, by definition of T, either σ or τ has left their initial state and thus $C_{(u)} \leq_{\mathrm{w}} A$ or $C_{(u)} \leq_{\mathrm{w}} A^{\mathsf{c}}$. In either case, we have $C \equiv_{\mathrm{w}} C_{(u)} \leq_{\mathrm{w}} A \oplus \varnothing$, and we are done with our initial goal, without needing to use $(C \cap [T], [T])$.

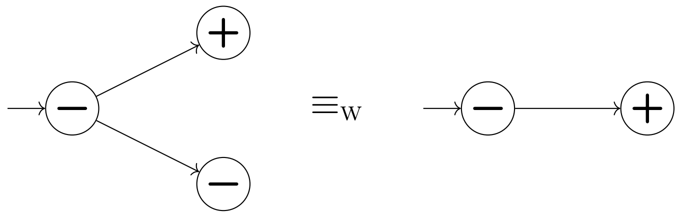
On the other hand, if such u always belong to T, we have that $\mathrm{Init}_C \subset T$. Towards a contradiction, suppose that $(C \cap [T], [T])$ is non-self-dual, that is, for all $x \in T$, there exists $u \subset x$ such that

$$\begin{aligned} (C \cap [T], [T])_{(u)} &<_{\mathrm{w}} (C \cap [T], [T]) \\ &\leq_{\mathrm{w}} C \cap [T] \\ &\leq_{\mathrm{w}} C. \end{aligned}$$

Therefore, when Player I is in charge of C and stays in T, there is some point where he is in charge of set weaker than C, and if he leaves T, he is also in charge of a weaker set. Hence Player I always ends up in charge of a set of a lower rank and C should be self-dual. □

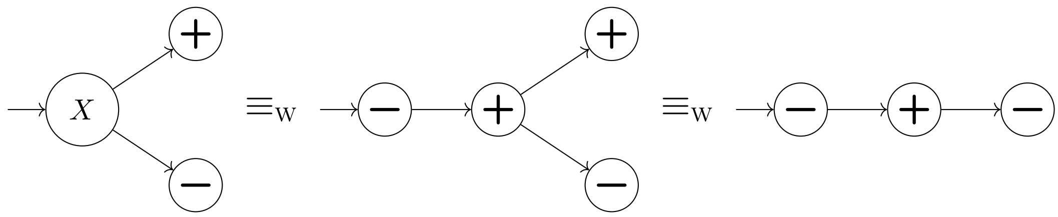
With the addition $\oplus$ between sets, we can, starting with the empty set, build sets of any finite rank, indeed a set of rank two is obtained as $\varnothing \oplus \varnothing$, a set of rank three is

$(\varnothing \oplus \varnothing) \oplus \varnothing$, and so on... However, we only visualize them as automatic sets, but we may want to have a more concrete description of the sets. Consider $X = \varnothing \oplus \varnothing$. One can see that its automatic set is equivalent to a simpler one, namely



Since traveling to the second ⊖ state doesn't change anything, as one could just stay in the first ⊖.

Therefore, $\varnothing \oplus \varnothing$ is some open set as we have already seen this automatic set. Now if we look at $\varnothing \oplus X$, we obtain



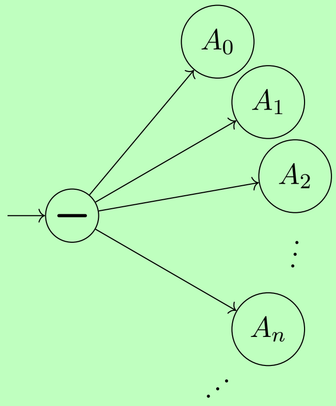
Since, for the sames reason as above, transitioning from the first ⊕ to the second doesn't help at all, so that we can convert any strategy using this transition to a strategy not using it.

If we label the transition with a one, that is, we change state every time (at most two) there is a 1 in the input, we obtain the set of sequences that contain exactly a one. This set can be seen as the intersection between the open set of sequences that contain at least one 1, and the closed set of sequences that contain at most one 1. In fact, one can show that this automatic set reduces exactly to sets that are the intersection of an open and a closed set, that is, the difference of two open sets.

Going further into the finite levels of the Wadge hierarchy, one can see that iterating n times the operation $X \mapsto \varnothing \oplus X$ is always equivalent to the automatic set consisting of $n+1$ states, alternating ⊖ and ⊕, starting with a ⊖. We could also show that those correspond to the difference of n open sets, thus establishing a link with Theorem 5.35.

Climbing up the hierarchy Now that we can reach all finite levels of the Wadge hierarchy, but more importantly, that we have an operation of addition that work at any level, we would only need an operation that allow us to handle the limit cases. However, it is not that simple as limit cases can be of a wide variety of types when looked at from our very countable point of view, that is, we always work with integers and sequences of integers. In fact we will not be able to define one operation to go above each limit ordinal. We start simple, with an operation $\sup^\omega$ to construct a set of a rank that is the limit of the rank of countably many sets.

Definition 5.25. Let $(A_n)_{n\in\omega}$ be a $<_\mathrm{W}$-increasing sequence of non-self-dual Borel sets. We define the set $\sup^\omega(A_n)$ as the language of



Proposition 5.26. Let $(A_n)_{n\in\omega}$ be a $<_\mathrm{W}$-increasing sequence of non-self-dual Borel sets, then the set $\sup^\omega(A_n)$ is non-self-dual.

Proof. Let $x \in \omega$ be any sequence whose run doesn't change state, that is it stays in $\circleddash$, then this sequence is idempotent, as at every moment one can still decide to stay in $\circleddash$ or change to any A_n. By Theorem 5.9 we conclude that $\sup^\omega(A_n)$ is non-self-dual. □

Theorem 5.27. Let $(A_n)_{n\in\omega}$ be a $<_\mathrm{W}$-increasing sequence of non-self-dual Borel sets.

$$\mathrm{rk}\left(\sup^\omega(A_n)\right) = \sup\left\{\mathrm{rk}(A_n) \mid n \in \omega\right\}.$$

Proof. Let $A = \sup^\omega(A_n)$. First, we clearly have $A_n \leq_\mathrm{W} A$, since a simple strategy in $G(A_n, A)$ consists of transitioning to the state A_n of the automatic set and then copying Player `I`. Thus,

$$\mathrm{rk}(A) \geq \sup \mathrm{rk}(A_n)$$

It remains to be shown that if B is a non-self-dual set such that $B <_\mathrm{W} A$, then there exists $n \in \omega$ such that $B \leq_\mathrm{W} A_n$. To that extent, let $x \in [\mathrm{Init}_B]$ be an infinite idempotent sequence for B. Since $B <_\mathrm{W} A$, we have both $B \leq_\mathrm{W} A$ and $B \leq_\mathrm{W} A^\mathsf{c}$. We consider A^c as the dual automatic set of A, that starts with a $\oplus$. Since Player `II` has a winning strategy in both $G(B, A)$ and $G(B, A^\mathsf{c})$, when Player `I` plays x, one of the two games must leave their initial state ($\circleddash$ or $\oplus$), as the disagree. Assume this happened in $G(B, A)$ since the other case is similar and let $k \in \omega$ be the moment when Player `II` leaves their initial state for some A_n (in the other case, it would be A_n^c).

Since Player `II` has a winning strategy, we have that $B_{(x\restriction_k)} \leq_\mathrm{W} A_n$, furthermore $B_{(x\restriction_k)} \equiv_\mathrm{W} B$ as x is idempotent, and therefore $\mathrm{rk}(B) \leq \mathrm{rk}(A_n)$ for some n.

Finally this shows that

$$\mathrm{rk}(A) = \{\mathrm{rk}(B) + 1 \mid B <_{\mathrm{w}} A\} \leq \{\mathrm{rk}(A_n) + 1 \mid n \in \omega\} \leq \sup \mathrm{rk}(A_n).$$

□

With this operation, we can now construct sets of any countable rank, as all countable ordinal are reachable from one, addition and countable union. We have thus realised the first ω_1 degrees of the Wadge hierarchy.

5.4 Construction of $\mathbf{\Delta}^0_3$ sets

With the operation of addition and countable union we can define a multiplication by countable ordinals on sets.

Definition 5.28. Let $A \subset \omega^\omega$ be a non-self-dual set and $\alpha \neq 0$ be a countable ordinal. We define $A \otimes \alpha$ by induction on α as the automatic set

$$A \otimes \alpha = \begin{cases} A & \text{if } \alpha = 1 \\ A \oplus (A \otimes \beta) & \text{if } \alpha = \beta + 1 \\ \sup^\omega\big((A \otimes \beta)_{\beta < \alpha}\big) & \text{otherwise.} \end{cases}$$

Proposition 5.29. Let $A \subset \omega^\omega$ be a non-self-dual set and $\alpha \neq 0$ be a countable ordinal, then

$$\mathrm{rk}(A \otimes \alpha) = \mathrm{rk}(A) \times \alpha.$$

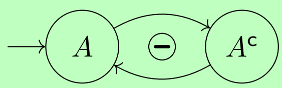
where $\times$ is the ordinal multiplication.

Proof. This follows by induction from the definition, Theorem 5.24 and Theorem 5.27. □

Notice that, if for a given ordinal α we expend recursively the definition, we always obtain an automatic set whose directed graph is a well founded tree, whose nodes are either $\ominus$, $\oplus$, $\textcircled{\scriptsize A}$ or $\textcircled{\scriptsize A^c}$. This fact will be crutial in the proof of Theorem 5.31.

We now define an automatic set that will correspond to multiplying the rank of a set by ω_1.

Definition 5.30. Let $A \subset \omega^\omega$ be a non-self-dual set, we define $A \otimes \omega_1$ as the language of



Theorem 5.31. Let $A \subset \omega^\omega$ be a non-self-dual set, then

$$\mathrm{rk}(A \otimes \omega_1) = \mathrm{rk}(A) \times \omega_1.$$

Proof. We first show that $\mathrm{rk}(A \otimes \omega_1) \geq \mathrm{rk}(A \otimes \alpha)$ for all $\alpha < \omega_1$. To that extent we construct a strategy in $G(A \otimes \alpha, A \otimes \omega_1)$, where both play in their respective automatic sets. We noticed that the automatic set for $A \otimes \alpha$ has states that are either $\ominus$, $\oplus$, $\textcircled{A}$ or $\textcircled{A^c}$. The strategy is thus:

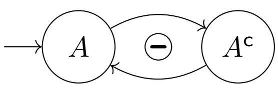
- If Player I changes state for $\oplus$ change state to go to a non-empty state, whether it is $\textcircled{A}$ or $\textcircled{A^c}$. If Player II is already in said state, change twice, for instance by playing (A, A^c).
- Similarly, if Player I changes state for $\ominus$, change state to go any state that is not ω^ω, changing twice if required.
- If Player I changes state for $\textcircled{A}$ or $\textcircled{A^c}$, change state to go to A or A^c respectively, changing twice if we already are in the desired state.
- If Player I is in either $\textcircled{A}$ or $\textcircled{A^c}$, and doesn't change state, play the same integer.
- if Player I is in $\oplus$, play one digit at a time of a sequence that belongs to the current state of Player II.
- Likewise, if Player I is in $\ominus$, play one digit at a time of a sequence that does not belong to the current state of Player II

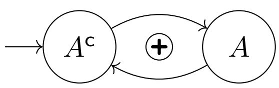
This startegy is winning, as Player I always stops changing state, since the tree of states in which he moves is well founded and thus Player II stops changing states at some point too. Furthermore, if x is the sequence played by Player I in his last state $S^{\mathtt{I}}$, and y is the sequence played by Player II in his last state $S^{\mathtt{II}}$, our construction implies that

$$x \in S^{\mathtt{I}} \iff y \in S^{\mathtt{II}}$$

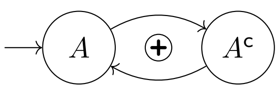
and thus Player II always wins.

For the other direction, let $C <_{\mathrm{w}} A \otimes \omega_1$ be a non-self-dual set. It suffices to show that $C \leq_{\mathrm{w}} A \otimes \alpha$ for some α countable. Since we have a strict reduction, we have that both $C \leq_{\mathrm{w}} A \otimes \omega_1$ and $C \leq_{\mathrm{w}} (A \otimes \omega_1)^c$. Therefore, Player II has a winning strategy when playing with both automatic set





And note that the rightmost automatic set is equivalent to



as Player II could start each of his plays with a transition to the other state and then play exactly the same. Since the first and third automatic sets are exactly the same but disagree when alternating infinitely many times, the strategies against C in their respective games never both alternate infinitely. By following the strategy that alternates the least, we can construct a strategy τ for the game $G(C, A \otimes \omega_1)$ that never alternates infinitely many times.

Let $X \subset \omega^{<\omega}$ be the set of sequences $s \in \omega^{<\omega}$ such that τ changes state when Player I plays s, that is, $\tau(s) \in \{A, A^{\mathsf{c}}\}$. Let $T \subset X^{<\omega}$ be the the tree that consists of sequences $(s_0, \dots, s_n)$ such that for all $i \leq n$, s_i is a prefix of s_n and s_i contains exactly $i + 1$ state transistions. This is indeed a tree as any prefix $(s_0, \dots, s_k) \subset (s_0, \dots, s_n)$ also satisfy the condition.

This tree is well founded, since if it had an infinite branch, it would mean that τ changes state infinitely many times along that play. Thus, let α be the height of the tree. We claim that $C \leq_{\mathrm{W}} A \otimes \alpha$. Let $s \in \omega^{<\omega}$ be an input of Player I and α_s be the height of s in the tree T. A strategy for Player II in $G(C, A \otimes \alpha)$ is to copy τ and follow the branch of $A \otimes \alpha$ that corresponds to $A \otimes \alpha_s$. $\square$

With the three set operations of addition, countable union and multiplication by ω_1, we can constructs sets that lay at every level of the Wadge hierarchy below $\omega_1^{\omega_1}$, which is the first ordinal closed under those three operations.

It turns out that those sets correspond exactly to the $\mathbf{\Delta}_3^0$ sets of the Borel hierarchy. And interesting parallel between the two hierarchies is through the operation of **difference** of sets and the difference hierarchy.

5.5 The difference hierarchy

Our last objective is to define an operation that builds a new set given an α-sequence of sets, for α an ordinal. The main idea is to generalise the difference of two sets $A \setminus B$ to multiple sets, and to do so into the transfinite. For finite sequences, a simple way to generalise is to build a set by repeatedly talking the difference with the previous one, so if we have $n + 1$ sets $(X_0, X_1, \dots, X_n)$, we can take their difference $D_{n+1}(X)$ as

$$D_{n+1}(X) = X_n \setminus (X_{n-1} \setminus (\cdots \setminus (X_1 \setminus X_0)))$$

Which is a very verbose way of talking about elements that, if say $n = 3$, are in X_3 but not X_2 except if they are in X_1 but not X_0. An other way of thinking is thus to first consider that $X_0 \subset X_1 \subset \cdots \subset X_n$ since elements of X_i that are not included in the next sets are not considered. Then, when n is odd, our difference can be written as

$$(X_n \setminus X_{n-1}) \cup \cdots \cup (X_3 \setminus X_2) \cup (X_1 \setminus X_0)$$

and if n is even,

$$(X_n \setminus X_{n-1}) \cup \cdots \cup (X_2 \setminus X_1) \cup X_0.$$

Here we see that the elements that belong to the difference are the element that are contained X_i but not in X_{i-1} when i has a different parity than n. Indeed, on Figure 9,

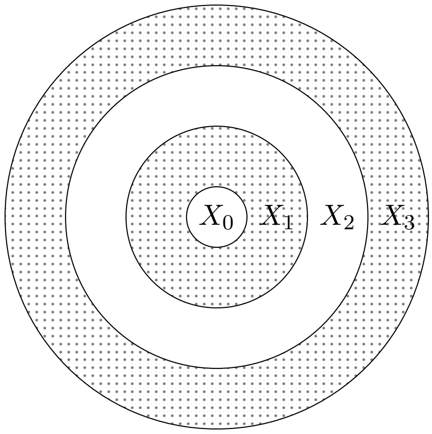


Figure 9: The difference of 4 sets is the dotted area.

the elements in the difference are precisely those that are first contained in X_1 and X_3, which have an opposite parity to four. The difference is then the set of all numbers such that the first set in which they appear has an index which has the opposite parity of n. This is precisely how we will generalise this idea, but we first need the notion of parity for all ordinals.

Definition 5.32. We say that an ordinal α is **even** if it can be written as $\alpha = \beta + 2k$ for β a limit ordinal or 0 and $k \in \omega$. An ordinal that is not even is **odd**.

If two ordinals α and β have opposite parity, we write $\alpha \not\equiv \beta \mod 2$.

Definition 5.33. Let α be an ordinal and $S = (S_0, S_1, \dots)$ be a $\subset$-increasing α-sequence of sets. We define the **difference** $D_\alpha(S)$ of S as the set of points x whose least β such that $x \in X_\beta$ has the opposite parity of α, that is:

$$D_\alpha(S) = \left\{ x \in \bigcup_{\beta<\alpha} S_\beta \;\middle|\; \exists \beta < \alpha \left(\begin{array}{c} x \in X_\beta \\ \wedge \\ \forall \gamma < \beta \; x \notin X_\gamma \\ \wedge \\ \beta \not\equiv \alpha \mod 2 \end{array} \right) \right\}$$

Definition 5.34. Let $X \subset \omega^\omega$ be a set, $\mathbf{\Gamma}$ be any class of sets and α an ordinal. We write $X \in D_\alpha(\mathbf{\Gamma})$ if there exists an α-sequence $S \in \mathbf{\Gamma}^\alpha$ such that $X = D_\alpha(S)$.

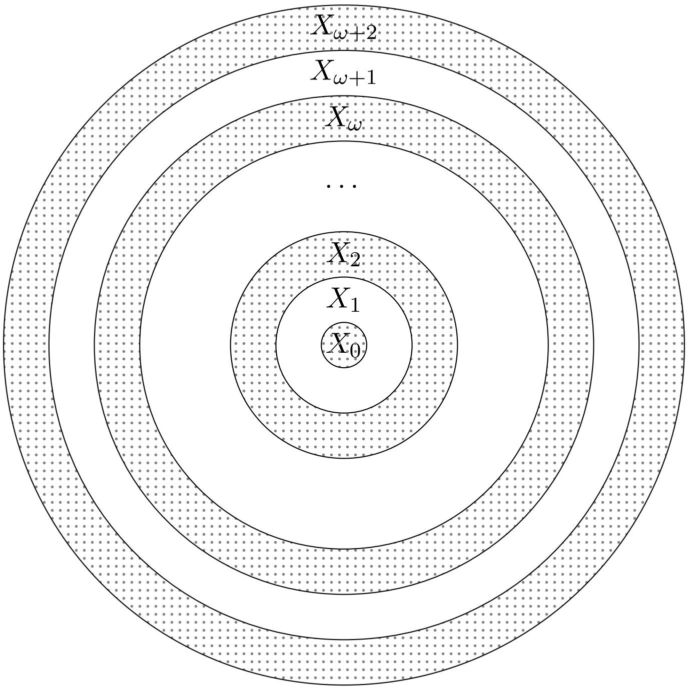


Figure 10: The difference of $\omega + 3$ sets.

The classes $D_\alpha(\mathbf{\Gamma})$ are called the **difference hierarchy** over $\mathbf{\Gamma}$.

Theorem 5.35 (Hausdorff)**.** A set $X \subset \omega^\omega$ is $\mathbf{\Delta}_2^0$ if and only if it is a countable difference of open sets.

$$\mathbf{\Delta}_2^0 = \bigcup_{\alpha<\omega_1} D_\alpha(\mathbf{\Sigma}_1^0)$$

In order to show this, we have many steps:

- For one direction, we prove by induction on α that $D_\alpha(\mathbf{\Sigma}_1^0) \subset \mathbf{\Delta}_2^0 2$
- For the other direction, we exhibit a $\mathbf{\Pi}_2^0$-complete set and a $\mathbf{\Sigma}_2^0$-complete set.
- Since both reduces to $A \in \mathbf{\Delta}_2^0$, we construct a game with a strategy that plays ones only finitely many times.
- From this strategy, we recover a well founded tree.
- On which we compute a new notion of height, that depends strongly on parity

- Which will give us a decomposition into a difference.

Proof. We start by showing by induction that a countable difference of open sets is $\mathbf{\Delta}^0_2$.

- For the base case, $\alpha = 0$, the difference of zero sets is the empty set, which is $\mathbf{\Delta}^0_2$.
- For the successor case, we take $X = (X_\xi)_{\xi<\alpha+1}$ an $(\alpha+1)$-sequence of open sets. Then

$$\begin{aligned} D_{\alpha+1}(X) &= X_\alpha \setminus D_\alpha(X{\restriction_\alpha}) \\ &= X_\alpha \cap D_\alpha(X{\restriction_\alpha})^{\mathsf{c}} \end{aligned}$$

  But X_α is an open set (thus $\mathbf{\Delta}^0_2$) and by the induction hypothesis, $D_\alpha(X{\restriction_\alpha})$ is $\mathbf{\Delta}^0_2$ and so is its complement, therefore, their intersection is also $\mathbf{\Delta}^0_2$.

- For the limit case, let $X = (X_\xi)_{\xi<\alpha}$ be an α sequence of open sets with α a limit ordinal. We first see that we can write the difference as a union of its partial differences (adding one ring at a time):

$$D_\alpha(X) = \bigcup_{\substack{\xi<\alpha \\ \xi \text{ even}}} D_\xi(X{\restriction_\xi})$$

  which is a countable union of $\mathbf{\Delta}^0_2$ by induction and thus $\mathbf{\Sigma}^0_2$. If we show that the complement of $D_\alpha(X)$ is also $\mathbf{\Sigma}^0_2$, we will be done. The complement of this difference is the union of all the rings that are not included in $D_\alpha(X)$, but those correspond exactly to the difference at odd steps, since α is even and thus the condition if inverted for odd ordinals. Therefore,

$$D_\alpha(X)^{\mathsf{c}} = \left(\bigcup_{\xi<\alpha} X_\xi\right)^{\mathsf{c}} \cup \bigcup_{\substack{\xi<\alpha \\ \xi \text{ odd}}} D_\xi(X{\restriction_\xi})$$

  which that it is again a countable union of $\mathbf{\Delta}^0_2$, because the X_ξ are open, so the first term is closed (thus $\mathbf{\Delta}^0_2$).

To show the reverse direction, we need to decompose any set $A \in \mathbf{\Delta}^0_2$ into a difference of countably many open sets.

Two complete sets. To that extent, we define two complete sets, B which is $\mathbf{\Sigma}^0_2$-complete

$$B = \{x \in \omega^\omega \mid \text{“}x \text{ has an even number of ones”}\}$$

and $C = B^{\mathsf{c}}$, the complement of B, which is $\mathbf{\Pi}^0_2$-complete

$$C = \{x \in \omega^\omega \mid \text{“}x \text{ has an odd number of ones or infinitely many”}\}$$

We can see easily that $B \in \mathbf{\Sigma}^0_2$, as we can write it as

$$\left\{ x \in \omega^\omega \;\middle|\; \exists n\ \forall m > n \ \begin{pmatrix} x(m) \neq 1 \\ \wedge \\ \#_1(x\restriction_n) \in 2\mathbb{N} \end{pmatrix} \right\}$$

and the condition is clopen as we know whether or not a sequence satisfies it only by looking at the first m digits. Moreover B is indeed $\mathbf{\Sigma}^0_2$-complete, as the set $X \subset \omega^\omega$ of the sequences that contain finitely many ones, which we have proven to be $\mathbf{\Sigma}^0_2$-complete in Proposition 3.8 reduces to B. To see that $X \leq_{\mathrm{W}} B$, we construct a winning strategy in the Wadge game $G(X, B)$: Player `II` plays the same integers as Player `I`, except when Player `I` plays a one, in which case he plays two. Therefore if Player `I` has a finite amount of ones, he has the double of that amount, which is even, if Player `I` doesn't, then Player `II` also has an infinite number of ones.

A winning strategy in $G(A, C)$ that never play infinitely many ones. Since B is $\mathbf{\Sigma}^0_2$-complete and A is $\mathbf{\Pi}^0_2$-complete, we have both $A \leq_{\mathrm{W}} B$ and $A \leq_{\mathrm{W}} C$ as the both reduce to A. Thus, let τ_B be a winning strategy for Player `II` in $G(A, B)$ and τ_C be a winning strategy in $G(A, C)$, we will construct a winning strategy τ in $G(A, C)$ such that Player `II` only plays finitely many ones. Notice that when Player `I` plays a sequence $x \in \omega^\omega$, there are two cases:

- Either $x \in A$, in which case, τ_B produces a sequence $y \in B$ which has an even number of ones
- Or $x \notin A$, in which case, τ_C produces a sequence $y \notin B$, so $y \in A$ which has an even number of ones.

Therefore there is always one of the two games in which Player `II` produces a sequence with finite ones. We would like to merge those two strategies by copying the one that currently has the least amount of ones. If for a given play $s \in \omega^{<\omega}$ of Player `I`, if $\tau_C(s)$ has the least ones, then we want to have the same parity as $\tau_C(s)$, so we play a one if needed, otherwise a zero, but if $\tau_C(s)$ has the least ones, we want to have the opposite parity so again we play a one match that parity. Formally, on a play $s^\frown j \in \omega^{<\omega}$ of Player `I`, let

$$b = \#_1(\tau_B(s^\frown j)) \qquad c = \#_1(\tau_C(s^\frown j)) \qquad n = \#_1(\tau(s))$$

be the number of ones played by τ_B and τ_C until now and the number of ones played by Player `II` until the last turn. The strategy τ is then defined as

$$\tau(s^\frown j) = \tau(s)^\frown \begin{cases} 42 & \text{if } b = c \\ (k - b + 1) \mod 2 & \text{if } b < c \\ (k - c) \mod 2 & \text{if } c < b \end{cases}$$

This strategy is winning as for any play $x \in \omega^\omega$ of Player `I`, there is an integer $N \in \omega$ such that one of the two strategies stop playing ones after N moves. If both do, we take we take the strategy that does so with the fewest ones, and we see that if it is τ_C, τ produces the same parity and therefore

$$x \in A \iff \tau_C(x) \in C \iff \tau(x) \in C.$$

However if it is τ_B, the parity is reversed and

$$x \in A \iff \tau_B(x) \in B \iff \tau(x) \notin B \iff \tau(x) \in B^{\mathsf{c}} = C.$$

Which concludes the fact that τ is a winning strategy that never plays infinitely many ones.

A well founded tree. Consider now the set X of finite sequences such that τ produces a one,

$$X = \left\{u \in \omega^{<\omega} \,\middle|\, \tau(u) = 1\right\}.$$

We construct the tree T on the alphabet X as the set of all sequences $\langle u_0, \ldots, u_n\rangle \in X^{<\omega}$ such that for all $i \leq n$, u_i is a prefix of u_n and contains exactly $i+1$ ones. This is indeed a tree, as any prefix $\langle u_0, \ldots, u_k\rangle \subset \langle u_0, \ldots, u_n\rangle \in T$ also satisfies the condition. The sequences in this tree can be seen as the steps where τ produces a one.

We now notice that this tree has no infinite branch, and thus is well founded because if $s \in [T]$ is infinite, it means that for all $n \in \omega$ $s(i) \in X$ and thus $\tau(s(i)) = 1$, therefore τ plays infinitely many ones when Player I plays $x = \bigcup_n s(n)$ which contradicts our assumption on τ.

A height depending on parity. For a given well founded tree T we recursively define its height on every node s as an ordinal $\alpha = \mathrm{ht}_T(s)$ in the following way:

- if $s \in T$ is a leaf and $\mathrm{lh}(s)$ is odd, we set $\mathrm{ht}_T(s) = 0$.
- if $s \in T$ is a leaf and $\mathrm{lh}(s)$ is even, we set $\mathrm{ht}_T(s) = 1$.
- if $s \in T$ is not a leaf, we set

  $$h = \sup\left\{\mathrm{ht}_T(s^\frown t) + 1 \,|\, t \in X \wedge s^\frown t \in T\right\} a,$$

  then $\mathrm{ht}_T(s) = h$ if h and $\mathrm{lh}(s)$ have opposite parity, otherwise we set $\mathrm{ht}_T(s) = h+1$.

Recovering the difference of open sets from the tree. Notice that $\mathrm{lh}(s)$ and $\mathrm{ht}_T(s)$ always have opposite parity, and therefore the height of the tree that we can define as $\mathrm{ht}_T(\varepsilon)$ is always odd. Notice also, that any sequence $s \in T$ corresponds to a finite play with exactly $\mathrm{lh}(s)$ ones, thus all those parity checks ensures that when there is an odd number of ones, that is, if the sequence is potentially in C, the height is even, otherwise it is odd.

Let α be the height of the tree T. For any $\xi < \alpha$, we define the set $\tilde{A}_\xi$ of plays of Player I such that the height of the corresponding node is ξ:

$$\tilde{A}_\xi = \{\mathrm{last}(s) \,|\, s \in T \wedge \mathrm{ht}_T(s) = \xi\} \subset X.$$

Here, A_0 is the set of to all the plays $s \in \omega^{<\omega}$ of Player I such that the corresponding height is 0, that is the sequences that Player II plays has an odd number of ones. Moreover, afterwards, Player II never plays ones again, so all of the sequences in $[s]$ are in A, i.e. $[s] \subset A$.

Similarly, A_1 is the set of plays $x \in \omega^{<\omega}$ of Player I such that

- either the Player II plays an even number of ones and never plays ones again, however Player I extends s.
- Player II can extend s into $t \in \tilde{A}_0$ in which case Player II will play one one afterwards but then never again.

More generally, if $s \in \tilde{A}_\xi$ and ξ is odd, either Player II never plays ones again and the sequence is in A or Player I extend s into $t \in \tilde{A}_{\xi'}$ with ξ' even. On the other hand, if $s \in \tilde{A}_\xi$ and ξ is even, either Player II never plays ones again and the sequence is not in A or Player I extend s into $t \in \tilde{A}_{\xi'}$ with ξ' odd.

This finally suggests that the sets which form the difference are

$$A_\xi = \bigcup_{s \in \tilde{A}_\xi} [s].$$

We now show by double inclusion that

$$A = D_\alpha\big((A_\xi)_{\xi<\alpha}\big).$$

Let $x \in D_\alpha\big((A_\xi)_{\xi<\alpha}\big)$, by the definition of the difference, and the fact that α is odd, we deduce that $\beta = \min\{\xi \mid x \in A_\xi\}$ is even. This means that the sequence on which the last one was played, $s = \bigcup\{s \subset x \mid \tau(s) = 1\}$ is in $\tilde{A}_\beta$, since β is even, $\tau(s)$ has an odd number of ones, thus $y = \bigcup_{t \subset x} \tau(t)$ has an odd number of ones and $y \in C$. But since τ is winning, $x \in A$ too.

Let $x \in A$ and $y \in \omega^\omega$ the sequence played by τ on input x. Since $x \in A$, we have $y \in C$ and y has an odd number of ones. Let $s \subset y$ the largest prefix that ends with a one. In the tree T, it corresponds to as sequence $t = \langle t_0, \ldots, t_k\rangle$ with $t_k = s$ such that $\mathrm{lh}(t)$ is odd, thus its height $\beta = \mathrm{ht}_T(t)$ is even. The last thing to show is that $\beta = \min\{\xi \mid x \in A_\xi\}$ as it would imply that $x \in D_\alpha\big((A_\xi)_{\xi<\alpha}\big)$, but this is clear as $x \in A_\beta$ and $x \in A_\xi$ for $\xi < \beta$ implies that there is a prefix $p \supset s$ of x that corresponds to a child of t in the tree T, and thus $\tau(p) = 1$, a contradiction with the maximality of s.

□

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