OCN-SIM Flex is part of OCN-SIM – an open-source collection of toolkits for the simulation and design of maritime systems and part of the project

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OCN-SIM FLEX

A tool for the approximate dynamic simulation of fully submerged flexible maritime structures

Theory Manual

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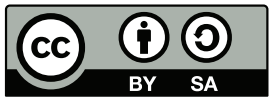

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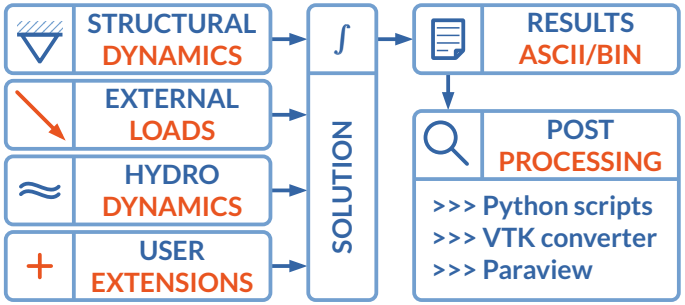

Figure 1 OCN-Sim Flex system architecture

OCN-SIM Flex is a **command line tool** for the approximate dynamic simulation of flexible maritime systems including simple rigid bodies. Examples are moorings, buoys, towed vehicles, nets and fishing gear.

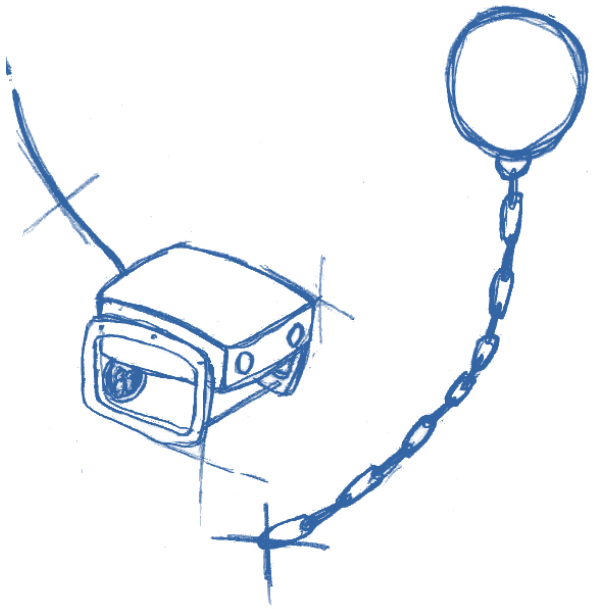

Figure 2 Example applications: A remotely operated vehicle and a buoy

Structural Elements: General Setup

Concerning structural mechanics, OCN-SIM Flex allows to create systems according to the topology show in figure 3.

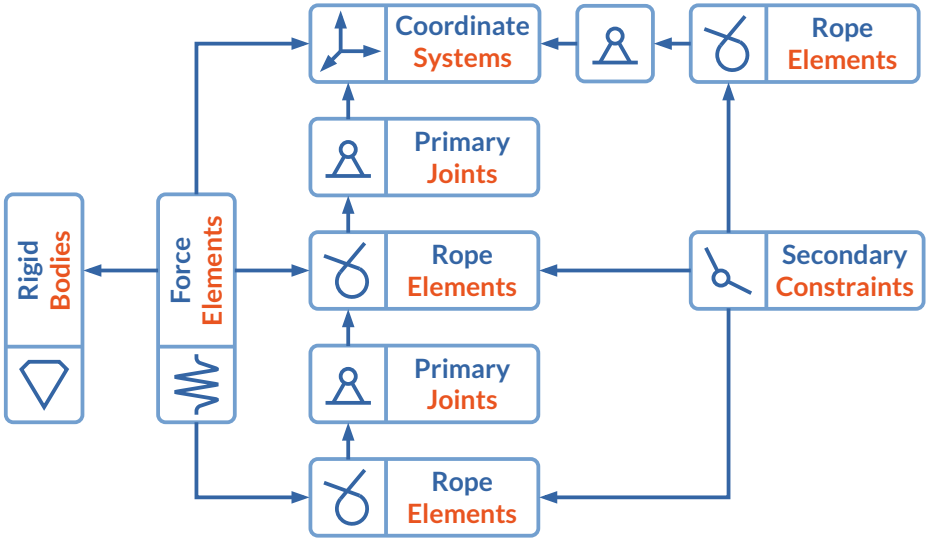

Figure 3 General setup of structural elements

A Note on XYZ–Euler Angles

In OCN-SIM Flex, XYZ-Euler angles are used to specify rotations. Thus, a rotation of a coordinate system $\mathcal{K}_i$ against a predecessor coordinate system $\mathcal{K}_p$ is described by three succeeding rotations about the x-, y- and z-axis as shown in the figure below. Remark: For rigid bodies rotations, the XYZ-Euler angles are internally converted to Euler-parameters, compare chapter "Structural Element: Rigid Body".

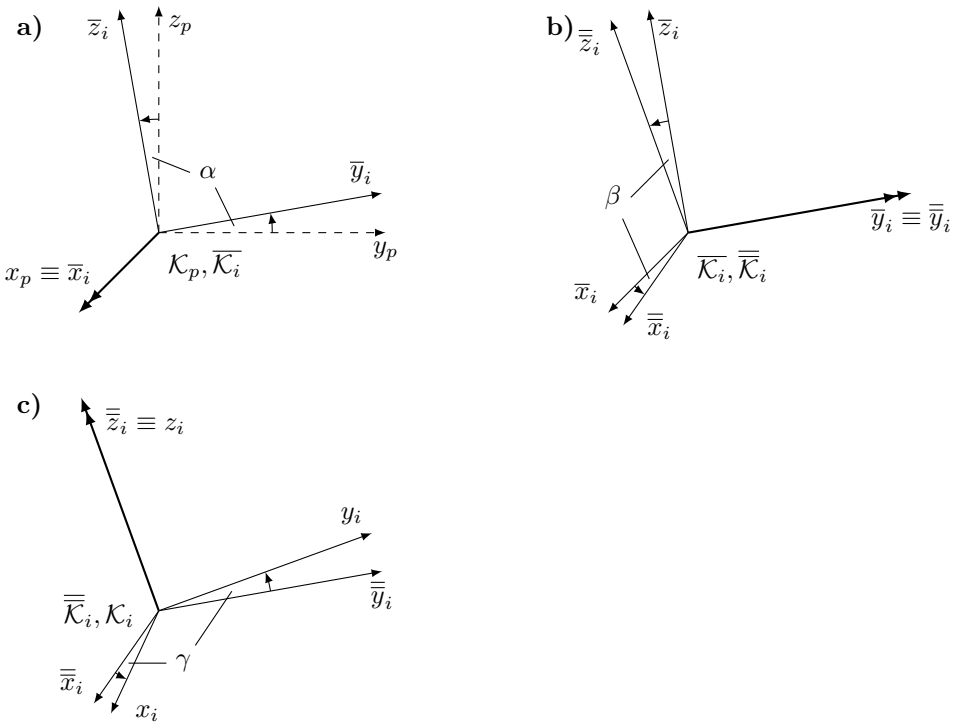

Figure 4 Order of rotation for XYZ-Euler angles. **a)** First rotation by α_i about x-axis. **b)** Second rotation by β_i about y-axis. **c)** Third rotation by γ_i about z-axis.

Symbols

$\boldsymbol{a}$	Acceleration vector of the complete system of nodes 8, 9
$\boldsymbol{f}^{\mathrm{e}}$	Vector of external forces acting on all nodes 9
$\boldsymbol{f}^{\mathrm{q,e}}$	Vector of generalised external forces acting on all nodes in terms of minimal coordinates $\boldsymbol{q}$ 9
$\boldsymbol{f}^{\mathrm{r}}$	Vector of reaction forces acting on all nodes 9
$\boldsymbol{f}_i^{\mathrm{r}}$	Vector of reaction forces acting on nodes i 13
$\bar{g}_{i_\mathrm{C}}$	Implicit constraint i 10
i_C	Counting index for implicit constraint equations 10, 11
$\boldsymbol{J}$	JACOBIAN-matrix of the complete system of nodes 8, 9
$\boldsymbol{J}_{ij}$	JACOBIAN-matrix of node i with respect to joint j 7, 8
$k1\,(i_\mathrm{C})$	Node number 1 in distance constraint $\bar{g}_{i_\mathrm{C}}$ 5, 10, 11, 14
$k2\,(i_\mathrm{C})$	Node number 2 in distance constraint $\bar{g}_{i_\mathrm{C}}$ 5, 10, 11, 14
$\boldsymbol{M}$	Mass matrix of the complete system of nodes 9
$\boldsymbol{M}^{\mathrm{q}}$	Generalised mass matrix of the complete system of nodes in terms of minimal coordinates $\boldsymbol{q}$ 9
n_N	Total number of nodes in the system 6–9
n_q	Total number of minimal coordinates 6
$\mathbb{P}_i$	Set of predecessor nodes of node i 6
$\bar{\mathbb{P}}_i$	Set of predecessor nodes of node i extended by node i itself 6
$\boldsymbol{q}$	System vector of joint coordinates 5, 6
$\ddot{\boldsymbol{q}}$	System vector of joint accelerations 9
$\dot{\boldsymbol{q}}$	System vector of joint velocities 8, 9
$\boldsymbol{r}$	Position vector of all nodes in coordinate system $\mathcal{K}_0$ 7
$\boldsymbol{r}_{\mathrm{g},i_\mathrm{C}}$	Difference vector of the positions of the two nodes pointing from $k1\,(i_\mathrm{C})$ to $k2\,(i_\mathrm{C})$ 10
$\boldsymbol{r}_i$	Position vector of node i in coordinate system $\mathcal{K}_0$ 6, 7
$\boldsymbol{u}_{\mathrm{g},i_\mathrm{C}}$	Direction of constraint i_C, i.e. the unit vector pointing from $k1\,(i_\mathrm{C})$ to $k2\,(i_\mathrm{C})$ 10, 11
$\boldsymbol{v}$	Velocity vector of the complete system of nodes in coordinate system $\mathcal{K}_0$ 8
$\boldsymbol{v}_i$	Velocity vector of node i in coordinate system $\mathcal{K}_0$ 7, 8

Physical model

- High tensile stiffness, negligibly low bending resistance
  ⇒ Idealisation as ideally flexible, inextensible continua
- Discrete depiction by massless, rigid bar elements interconnected by rotary joints
- Discrete lumped masses at ends of bar elements
- Longitudinal axis of the bar element: local z-axis
- No rotation about local z-axis
- No explicitly time-dependant displacements (e.g. actuators)

Geometric definitions

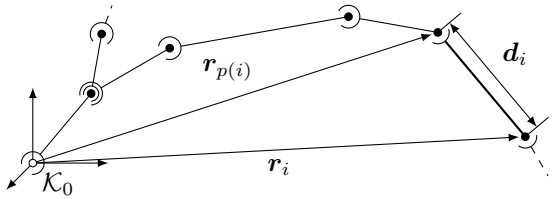

Figure 5 Position $\boldsymbol{r}_i$ and element length $\boldsymbol{d}_i$ of node i and position $\boldsymbol{r}_i$ of predecessor $p(i)$ of node i

Lagrangian Dynamics – Non-recursive Formulation

Method

- Integration of equations of motion in minimal coordinates
- Explicit constraint equations ⇒ Constraint equations do not need to be integrated and are always fulfilled
- Error resulting from numerical integration only in dynamic behaviour

Model

- Equations of motion derived using Lagrangian dynamics and a set of n_q minimal coordinates

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_{n_\mathrm{q}} \end{bmatrix} \tag{1}$$

- Constraints given in explicit form as

$$\boldsymbol{r} = \begin{bmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_{n_\mathrm{N}} \end{bmatrix} = \boldsymbol{r}\left(\boldsymbol{q}, t\right) \tag{2}$$

  with the position vector of node i $\boldsymbol{r}_i$ and the total number of nodes n_N.

- Scleronomous system, i.e. the position vector only depends on the generalized coordinates $\boldsymbol{q}$ and not on time

$$\boldsymbol{r} = \boldsymbol{r}\left(\boldsymbol{q}\right) \tag{3}$$

- Branched multibody system with tree structure
- Kinematic loops by introduction of additional loop-closing constraints and associated reaction forces
- Loop-closing constraints implemented as PID force elements
- Rotation about the local z-axis is not considered
- Minimal coordinates: relative XY-Euler joint angles α_i and β_i of body i

Definitions

Let $\mathbb{P}_i$ be the set of predecessor nodes of node i, $\bar{\mathbb{P}}_i = \{\mathbb{P}_i, i\}$ the set of predecessor nodes extended by node i itself, ${}^{i}\boldsymbol{d}_i$ the displacement vector of node i with respect to its joint in the local coordinate system $\mathcal{K}_i$, $\boldsymbol{q}_i = [\alpha_i\ \beta_i]^\mathrm{T}$ the vector containing the relative joint XY-Euler angles α_i as well as β_i and ${}^{0k}\boldsymbol{T}$ the transformation matrix from $\mathcal{K}_i$ to the reference system $\mathcal{K}_0$.

Positions

The position ${}^{0}\boldsymbol{r}_i$ of node i evaluated in the global coordinate system $\mathcal{K}_0$ is the sum of all displacement vectors of it's predecessor nodes $\mathbb{P}_i$:

$${}^{0}\boldsymbol{r}_i = \sum_{k \in \bar{\mathbb{P}}_i} {}^{0k}\boldsymbol{T}\ {}^{k}\boldsymbol{d}_k \tag{4}$$

Gathered together, the vectors ${}^0\boldsymbol{r}_i$ form the position vector of all nodes in the system

$$ {}^0\boldsymbol{r} = \begin{bmatrix} {}^0\boldsymbol{r}_1 \\ \vdots \\ {}^0\boldsymbol{r}_{n_N} \end{bmatrix} . \tag{5} $$

Velocities

Applying the chain rule of differentiation, the velocity of node i is retrieved as

$$ \begin{aligned} {}^0\boldsymbol{v}_i &= \frac{\mathrm{d}\,{}^0\boldsymbol{r}_i}{\mathrm{d}\,t} = \sum_{k\in\bar{\mathbb{P}}_i} \frac{\mathrm{d}\,{}^{0k}\boldsymbol{T}\;{}^k\boldsymbol{d}_k}{\mathrm{d}\,t} = \sum_{k\in\bar{\mathbb{P}}_i}\sum_{j\in\bar{\mathbb{P}}_i} \frac{\partial {}^{0k}\boldsymbol{T}\;{}^k\boldsymbol{d}_k}{\partial\alpha_j}\dot{\alpha}_j + \frac{\partial {}^{0k}\boldsymbol{T}\;{}^k\boldsymbol{d}_k}{\partial\beta_j}\dot{\beta}_j \\ &= \sum_{j\in\bar{\mathbb{P}}_i}\sum_{k\in\bar{\mathbb{P}}_i} \frac{\partial {}^{0k}\boldsymbol{T}\;{}^k\boldsymbol{d}_k}{\partial\alpha_j}\dot{\alpha}_j + \frac{\partial {}^{0k}\boldsymbol{T}\;{}^k\boldsymbol{d}_k}{\partial\beta_j}\dot{\beta}_j \,. \end{aligned} \tag{6} $$

Since ${}^k\boldsymbol{d}_k = const.$, the derivatives simplify to

$$ {}^0\boldsymbol{v}_i = \sum_{j\in\bar{\mathbb{P}}_i}\sum_{k\in\bar{\mathbb{P}}_i} \frac{\partial {}^{0k}\boldsymbol{T}}{\partial\alpha_j}{}^k\boldsymbol{d}_k\dot{\alpha}_j + \frac{\partial {}^{0k}\boldsymbol{T}}{\partial\beta_j}{}^k\boldsymbol{d}_k\dot{\beta}_j \tag{7} $$

Moreover, the derivative becomes $\boldsymbol{0}_{(3)}$ if the position of node k does not depend on the state of joint j. Since this is only the case if k is a predecessor of j, i.e. $k \in \mathbb{P}_j$, the subset of predecessor nodes of node i depending on the state of joint j is introduced as

$$ \bar{\mathbb{P}}_{ij} = \bar{\mathbb{P}}_i \setminus \mathbb{P}_j = \{j,\ \ldots,\ p(i),\ i\} \,, \tag{8} $$

so that the above relationship can be rewritten as

$$ {}^0\boldsymbol{v}_i = \sum_{j\in\bar{\mathbb{P}}_i}\sum_{k\in\bar{\mathbb{P}}_{ij}} \frac{\partial {}^{0k}\boldsymbol{T}}{\partial\alpha_j}{}^k\boldsymbol{d}_k\dot{\alpha}_j + \frac{\partial {}^{0k}\boldsymbol{T}}{\partial\beta_j}{}^k\boldsymbol{d}_k\dot{\beta}_j = \sum_{j\in\bar{\mathbb{P}}_i}\sum_{k\in\bar{\mathbb{P}}_{ij}} \boldsymbol{v}_{ij} \,. \tag{9} $$

The velocity component $\boldsymbol{v}_{ij}$ of node i resulting from the velocity of joint $j \in \bar{\mathbb{P}}_i$ can then be specified in matrix form as

$$ \begin{aligned} \boldsymbol{v}_{ij} &= \begin{bmatrix} \sum_{k\in\bar{\mathbb{P}}_{ij}} \frac{\partial {}^{0k}\boldsymbol{T}}{\partial\alpha_j}{}^k\boldsymbol{d}_k & \sum_{k\in\bar{\mathbb{P}}_{ij}} \frac{\partial {}^{0k}\boldsymbol{T}}{\partial\beta_j}{}^k\boldsymbol{d}_k \end{bmatrix} \begin{bmatrix} \dot{\alpha}_j \\ \dot{\beta}_j \end{bmatrix} \quad , \quad j \in \bar{\mathbb{P}}_i \\ \boldsymbol{v}_{ij} &= \qquad\qquad \boldsymbol{J}_{ij} \qquad\qquad\qquad \dot{\boldsymbol{q}}_j \end{aligned} \tag{10} $$

with the JACOBIAN-matrix $\boldsymbol{J}_{ij}$ of node i with respect to joint j

$$ \boldsymbol{J}_{ij} = \begin{bmatrix} \boldsymbol{J}_{\alpha,ij} & \boldsymbol{J}_{\beta,ij} \end{bmatrix} = \begin{bmatrix} \sum_{k\in\bar{\mathbb{P}}_{ij}} \frac{\partial {}^{0k}\boldsymbol{T}}{\partial\alpha_j}{}^k\boldsymbol{d}_k & \sum_{k\in\bar{\mathbb{P}}_{ij}} \frac{\partial {}^{0k}\boldsymbol{T}}{\partial\beta_j}{}^k\boldsymbol{d}_k \end{bmatrix} . \tag{11} $$

Introducing the joint transformation matrix

$$ {}^i\boldsymbol{S} \equiv {}^{p(i),i}\boldsymbol{T} \tag{12} $$

and it's derivatives

$$ {}^i\boldsymbol{S}_\alpha = \frac{\partial {}^i\boldsymbol{S}}{\partial\alpha_i} \tag{13a} $$

$$ {}^i\boldsymbol{S}_\beta = \frac{\partial {}^i\boldsymbol{S}}{\partial\beta_i} \tag{13b} $$

the transformation matrix ${}^{0k}\boldsymbol{T}$ can be split up into the succeeding transformations

$$ {}^{0k}\boldsymbol{T} = {}^{0,p(j)}\boldsymbol{T}\;{}^j\boldsymbol{S}\;{}^{jk}\boldsymbol{T} \tag{14} $$

and the derivatives occuring in the above equations can be rewritten as

$$ \frac{\partial {}^{0k}\boldsymbol{T}}{\partial\alpha_j} = {}^{0,p(j)}\boldsymbol{T}\;{}^j\boldsymbol{S}_\alpha\;{}^{jk}\boldsymbol{T} \tag{15a} $$

$$ \frac{\partial {}^{0k}\boldsymbol{T}}{\partial\beta_j} = {}^{0,p(j)}\boldsymbol{T}\;{}^j\boldsymbol{S}_\beta\;{}^{jk}\boldsymbol{T} \,. \tag{15b} $$

Substitution into (11) leads to:

$$ \boldsymbol{J}_{\alpha/\beta,ij} = \sum_{k\in\bar{\mathbb{P}}_{ij}} {}^{0,p(j)}\boldsymbol{T}\;{}^j\boldsymbol{S}_{\alpha/\beta}\;{}^{jk}\boldsymbol{T}^k\boldsymbol{d}_k = \sum_{k\in\bar{\mathbb{P}}_{ij}} {}^{0,p(j)}\boldsymbol{T}\;{}^j\boldsymbol{S}_{\alpha/\beta}\;{}^j\boldsymbol{d}_k \,. \tag{16} $$

Since the term ${}^{0,p(j)}\boldsymbol{T}\;{}^j\boldsymbol{S}_{\alpha/\beta}$ does not depend on index k, it can be factored out of the sum, so that we get

$$ \boldsymbol{J}_{\alpha/\beta,ij} = {}^{0,p(j)}\boldsymbol{T}\;{}^j\boldsymbol{S}_{\alpha/\beta} \sum_{k\in\bar{\mathbb{P}}_{ij}} {}^j\boldsymbol{d}_k \,. \tag{17} $$

The sum in the equation above is can be identified as the vector pointing from the predecessor of node j to node i. Thus, introducing

$$ \boldsymbol{d}_{i,p(j)} = \boldsymbol{r}_i - \boldsymbol{r}_{p(j)} \tag{18} $$

as a shorthand, the masspoint's JACOBIAN-matrix with respect to joint $j \in \bar{\mathbb{P}}_i$ from (11) can now be specified in compact form as

$$\boldsymbol{J}_{ij} = {}^{0,p(j)}\boldsymbol{T} \begin{bmatrix} {}^{j}\boldsymbol{S}_{\alpha}{}^{j}\boldsymbol{d}_{i,p(j)} & {}^{j}\boldsymbol{S}_{\beta}{}^{j}\boldsymbol{d}_{i,p(j)} \end{bmatrix}, \quad j \in \bar{\mathbb{P}}_i \,. \tag{19}$$

Finally, the sub-JACOBIAN-matrices $\boldsymbol{J}_{ij},\ j \in \bar{\mathbb{P}}_i$ as well as the associated joint velocities $\boldsymbol{q}_{ij},\ j \in \bar{\mathbb{P}}_i$ can be arranged into the complete set of joint velocities $\boldsymbol{q}$ so as to derive a projection of the complete set of velocities onto the CARTESIAN velocities of node i.

$$\begin{aligned} {}^{0}\boldsymbol{v}_i &= \begin{bmatrix} \boldsymbol{J}_{i1} & \cdots & \boldsymbol{J}_{i,n_{\mathrm{N}}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}_1 \\ \vdots \\ \dot{\boldsymbol{q}}_{n_{\mathrm{N}}} \end{bmatrix} \\ {}^{0}\boldsymbol{v}_i &= \boldsymbol{J}_i \qquad\qquad \dot{\boldsymbol{q}} \end{aligned} \tag{20}$$

Here all submatrices $\boldsymbol{J}_{ij}$ become $\boldsymbol{0}_{(3,2)}$ if $j \not\in \bar{\mathbb{P}}_i$ and (19) needs to be extended to

$$\boldsymbol{J}_{ij} = \begin{cases} {}^{0,p(j)}\boldsymbol{T} \begin{bmatrix} {}^{j}\boldsymbol{S}_{\alpha}{}^{j}\boldsymbol{d}_{i,p(j)} & {}^{j}\boldsymbol{S}_{\beta}{}^{j}\boldsymbol{d}_{i,p(j)} \end{bmatrix} & \text{if } j \in \bar{\mathbb{P}}_i \\ \boldsymbol{0}_{(3,2)} & \text{otherwise,} \end{cases} \tag{21}$$

Gathered together, the vectors ${}^{0}\boldsymbol{v}_i$ form the velocity vector of all nodes in the system

$$\begin{aligned} \begin{bmatrix} {}^{0}\boldsymbol{v}_1 \\ \vdots \\ {}^{0}\boldsymbol{v}_{n_{\mathrm{N}}} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{J}_1 \\ \vdots \\ \boldsymbol{J}_{n_{\mathrm{N}}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}_1 \\ \vdots \\ \dot{\boldsymbol{q}}_{n_{\mathrm{N}}} \end{bmatrix} \\ {}^{0}\boldsymbol{v} &= \boldsymbol{J} \qquad \dot{\boldsymbol{q}} \end{aligned} \tag{22}$$

with the JACOBIAN-matrix $\boldsymbol{J}$ of all nodes

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_1 \\ \vdots \\ \boldsymbol{J}_{n_{\mathrm{N}}} \end{bmatrix} . \tag{23}$$

Physical interpretation

Consider the velocity component $\boldsymbol{v}_{ij}$ of node i due to the angular velocities of joint $j \in \bar{\mathbb{P}}_i$ as given by (10) together with the definition of the according JACOBIAN-matrix from (19):

$$\begin{aligned} {}^{0}\boldsymbol{v}_{ij} &= {}^{0,p(j)}\boldsymbol{T} \begin{bmatrix} {}^{j}\boldsymbol{S}_{\alpha}{}^{j}\boldsymbol{d}_{i,p(j)} & {}^{j}\boldsymbol{S}_{\beta}{}^{j}\boldsymbol{d}_{i,p(j)} \end{bmatrix} \begin{bmatrix} \dot{\alpha}_j \\ \dot{\beta}_j \end{bmatrix} \\ &= {}^{0,p(j)}\boldsymbol{T} \left({}^{j}\boldsymbol{S}_{\alpha}\dot{\alpha} + {}^{j}\boldsymbol{S}_{\beta}\dot{\beta} \right) {}^{j}\boldsymbol{d}_{i,p(j)} \\ &= {}^{0,p(j)}\boldsymbol{T}\ {}^{j}\dot{\boldsymbol{S}}\ {}^{j}\boldsymbol{d}_{i,p(j)} \equiv {}^{0,p(j)}\boldsymbol{T}\ {}^{p(j),j}\dot{\boldsymbol{T}}\ {}^{j}\boldsymbol{d}_{i,p(j)} \,. \end{aligned} \tag{24}$$

Inserting the product ${}^{p(j),j}\boldsymbol{T}^{\mathrm{T}}\ {}^{p(j),j}\boldsymbol{T} = \boldsymbol{I}_{(3,3)}$ in the above equation results in

$${}^{0}\boldsymbol{v}_{ij} = {}^{0,p(j)}\boldsymbol{T}\ {}^{p(j),j}\dot{\boldsymbol{T}}\ \underbrace{{}^{p(j),j}\boldsymbol{T}^{\mathrm{T}}\ {}^{p(j),j}\boldsymbol{T}}_{\boldsymbol{I}_{(3,3)}}\ {}^{j}\boldsymbol{d}_{i,p(j)} \,. \tag{25}$$

Here, the product ${}^{p(j),j}\dot{\boldsymbol{T}}\ {}^{p(j),j}\boldsymbol{T}^{\mathrm{T}}$ can be identified as the skew symmetric matrix ${}^{p(j)}\tilde{\boldsymbol{\omega}}_{j,p(j)}$ of angular velocities of j with respect to $p(j)$ in coordinate system $\mathcal{K}_{p(j)}$ (compare e.g. [5]) as

$${}^{p(j),j}\dot{\boldsymbol{T}}\ {}^{p(j),j}\boldsymbol{T}^{\mathrm{T}} = {}^{p(j)}\tilde{\boldsymbol{\omega}}_{j,p(j)} \,. \tag{26}$$

Substitution of this relationship and insertion of the product ${}^{0,p(j)}\boldsymbol{T}^{\mathrm{T}}\ {}^{0,p(j)}\boldsymbol{T} = \boldsymbol{I}_{(3,3)}$ into (25) leads to

$${}^{0}\boldsymbol{v}_{ij} = {}^{0,p(j)}\boldsymbol{T}\ {}^{p(j)}\tilde{\boldsymbol{\omega}}_{j,p(j)}\ \underbrace{{}^{0,p(j)}\boldsymbol{T}^{\mathrm{T}}\ {}^{0,p(j)}\boldsymbol{T}}_{\boldsymbol{I}_{(3,3)}}\ {}^{p(j),j}\boldsymbol{T}\ {}^{j}\boldsymbol{d}_{i,p(j)} \,, \tag{27}$$

which can be gathered to

$${}^{0}\boldsymbol{v}_{ij} = {}^{0}\tilde{\boldsymbol{\omega}}_{j,p(j)}\ {}^{0}\boldsymbol{d}_{i,p(j)} \,. \tag{28}$$

Consequently, it can be found from this relationship, that the velocity component $\boldsymbol{v}_{ij}$ of node i due to the angular velocities of joint $j \in \bar{\mathbb{P}}_i$ can be interpreted as the cross product of the angular velocity of node j with respect to it's predecessor $p(j)$ and distance from joint j to node i. This fact is illustrated in figure 6 below.

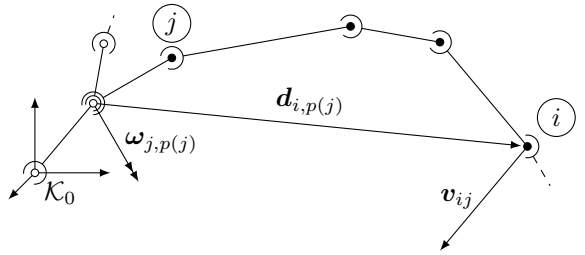

Figure 6 Velocity component $\boldsymbol{v}_{ij}$

Accelerations

Differentiating (22) with respect to time leads to the system's vector of accelerations

$${}^{0}\boldsymbol{a} = \frac{\mathrm{d}\,{}^{0}\boldsymbol{v}}{\mathrm{d}\,t} = \boldsymbol{J}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}\dot{\boldsymbol{q}} \,. \tag{29}$$

The derivation of the JACOBIAN with respect to time leads to quadratic terms in the joint velocities $\dot{\boldsymbol{q}}$ in the last term in the above equation. However, when ignoring high frequent vibrations induced e.g. by vortexes, the time derivatives of the joint angles are typically very small, i.e. $\dot{\boldsymbol{q}} \ll 1$. Consequently the resulting quadratic velocity terms can be neglected resulting in

$$ {}^0\boldsymbol{a} = \boldsymbol{J}\ddot{\boldsymbol{q}} + \cancelto{\boldsymbol{0}_{(3)}}{\dot{\boldsymbol{J}}\dot{\boldsymbol{q}}} = \boldsymbol{J}\ddot{\boldsymbol{q}} . \tag{30} $$

Equations of Motion

Applying the principle of virtual power (compare e.g. [5]), the equations of motion of all nodes can be derived as

$$ \begin{bmatrix} m_1\boldsymbol{I}_{(3,3)} & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & m_{n_\mathrm{N}}\boldsymbol{I}_{(3,3)} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{n_\mathrm{N}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^e_1 \\ \vdots \\ \boldsymbol{f}^e_{n_\mathrm{N}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{f}^r_1 \\ \vdots \\ \boldsymbol{f}^r_{n_\mathrm{N}} \end{bmatrix} $$

$$ \boldsymbol{M} \qquad \boldsymbol{a} \quad = \quad \boldsymbol{f}^\mathrm{e} \quad + \quad \boldsymbol{f}^\mathrm{r} . \tag{31} $$

Note that the upper left index indicating the coordinate system is omitted, since the following derivations are valid in any arbitrary, yet consistent, coordinate system. Substitution of the CARTESIAN accelerations depending on the accelerations of the joint states from equation (30) in the above equation and premultiplication with the transposed JACOBIAN leads to

$$ \begin{aligned} \boldsymbol{J}^\mathrm{T}\boldsymbol{M}\boldsymbol{J}\ddot{\boldsymbol{q}} &= \boldsymbol{J}^\mathrm{T}\boldsymbol{f}^\mathrm{e} + \cancelto{\boldsymbol{0}}{\boldsymbol{J}^\mathrm{T}\boldsymbol{f}^\mathrm{r}} \\ \boldsymbol{M}^\mathrm{q}\boldsymbol{a} &= \boldsymbol{f}^{\mathrm{q,e}} \end{aligned} \tag{32} $$

because the reaction forces are perpendicular to the constraint manifold, which is defined by the columns of the JACOBIAN-matrix. The system's generalised mass matrix

$$ \boldsymbol{M}^\mathrm{q} = \boldsymbol{J}^\mathrm{T}\boldsymbol{M}\boldsymbol{J} \tag{33} $$

as well as the vector of generalised external forces

$$ \boldsymbol{f}^{\mathrm{q,e}} = \boldsymbol{J}^\mathrm{T}\boldsymbol{f}^\mathrm{e} \tag{34} $$

appearing in the above equations can be interpreted as projections of the respective inertia and external forces to the constraint manifold.

Reconstructed Reaction Forces Formulation (RRF)

Classical methods – Lagrangian dynamics

- Equations of motion either based on explicit constraint equations and minimal coordinates or implicit constraint equations and redundant coordinates
- Depending on choice of formulation exact solution of constraints (explicit equations) or violation of constraints due to numerical integration of constraints (implicit constraints)

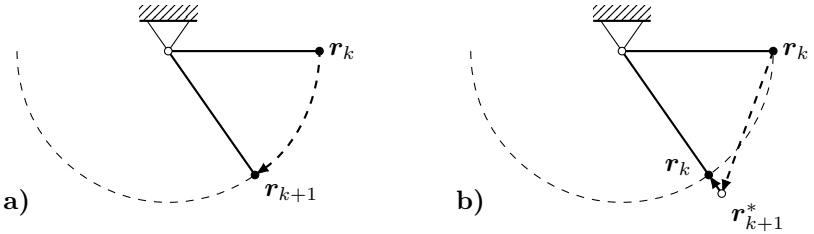

Figure 7 Example: Pendulum. **a)** Exact solution of explicit constraints equations. **b)** Violation of constraints due to numerical integration of implicit constraint equations. Violation corrected by projection.

Reconstructed reaction forces formulation

- Integration using projection-based solvers with two steps
- Based on extensions of algorithms presented in, e.g., [2] and [1]
- First published in [4] and extended in [3]
- Tw-step predictor-corrector algorithm
- **I – Predictor step**
  - Neglect reaction forces, accelerations solely due to external forces
  - Accelerations, velocities and positions dissatisfy constraints
  - Project positions to constraint manifold

- **II – Corrector step**
  - Reconstruct averaged reaction forces based on projected position
  - Apply trapezoidal rule based on external forces at end of timestep and averaged reaction forces
  - Project positions to constraint manifold

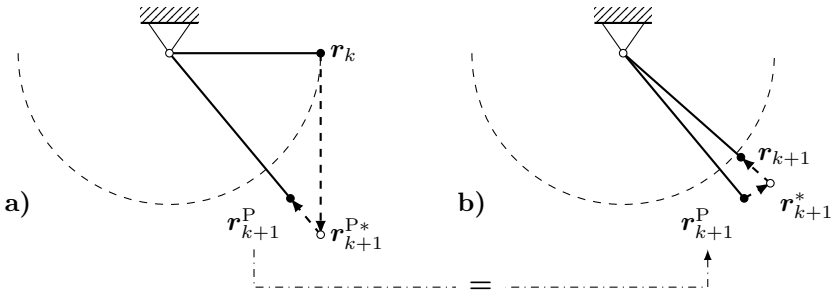

Figure 8 Example: Pendulum. **a)** Predictor step. **b)** Corrector step.

Model

- Equations of motion based on Lagrangian dynamics and dependant coordinates ⇒ Primary constraints given in implicit form as

$$\bar{\boldsymbol{g}}\left(\boldsymbol{r},t\right)=\begin{bmatrix}\bar{g}_1\left(\boldsymbol{r},t\right)\\ \vdots\\ \bar{g}_{n_{\mathrm{C}}}\left(\boldsymbol{r},t\right)\end{bmatrix}=\boldsymbol{0} \tag{35}$$

- Scleronomous system, i.e. the position vector only depends on the generalized coordinates $\boldsymbol{q}$ and not on time

$$\bar{\boldsymbol{g}}\left(\boldsymbol{r}\right)=\boldsymbol{0} \tag{36}$$

- Branched multibody system with tree structure
- Kinematic loops by introduction of additional loop-closing constraints and associated reaction forces
- Mathematically identical treatment of primary and loop-closing constraints
- Rotation about the local z-axis is not considered
- Minimal coordinates: relative XY-Euler joint angles α_i and β_i of body i

Projection of positions

The *Relaxation* algorithm as described in [2] or [1] projects the positions $\boldsymbol{r}^*$ violating the constraint equations back towards the constraint manifold, thus that the projected positions $\boldsymbol{r}$ approximately fullfill the constraint condition $\boldsymbol{g}(\boldsymbol{r})\approx\boldsymbol{0}$. In order to express this algorithm mathematically, the operator

$$\boldsymbol{r}=\mathrm{relax}(\boldsymbol{r}^*) \tag{37}$$

is introduced. Consider the i_{C}th distance constraint $\bar{g}_{i_{\mathrm{C}}}\left(\boldsymbol{r}\right)$ restricting the positions

$$\boldsymbol{r}_{1,i_{\mathrm{C}}}\equiv\boldsymbol{r}_{k1(i_{\mathrm{C}})}\left(\boldsymbol{r}\right) \tag{38a}$$
$$\boldsymbol{r}_{2,i_{\mathrm{C}}}\equiv\boldsymbol{r}_{k2(i_{\mathrm{C}})}\left(\boldsymbol{r}\right) \tag{38b}$$

of two nodes $k1\left(i_{\mathrm{C}}\right)$ and $k2\left(i_{\mathrm{C}}\right)$ to be at given a distance $l_{i_{\mathrm{C}}}$. Here, the difference of the positions of the nodes

$$\boldsymbol{r}_{\mathrm{g},i_{\mathrm{C}}}=\boldsymbol{r}_{2,i_{\mathrm{C}}}-\boldsymbol{r}_{1,i_{\mathrm{C}}}\,. \tag{39}$$

is defined to be the vector pointing from node $k1\left(i_{\mathrm{C}}\right)$ to $k2\left(i_{\mathrm{C}}\right)$. Also, the so-called *direction* of the constraint

$$\boldsymbol{u}_{\mathrm{g},i_{\mathrm{C}}}=\frac{\boldsymbol{r}_{\mathrm{g},i_{\mathrm{C}}}}{|\boldsymbol{r}_{\mathrm{g},i_{\mathrm{C}}}|} \tag{40}$$

is introduced as the unity-vector pointing in the same direction as $\boldsymbol{r}_{\mathrm{g},i_{\mathrm{C}}}$.

The distance between the nodes

$$d_{i_{\mathrm{C}}}\left(\boldsymbol{r}\right)\equiv|\boldsymbol{r}_{\mathrm{g},i_{\mathrm{C}}}|=\sqrt{\boldsymbol{r}_{\mathrm{g},i_{\mathrm{C}}}{}^2} \tag{41}$$

must then equal the specified length $l_{i_{\mathrm{C}}}$, so that the condition

$$\bar{g}_{i_{\mathrm{C}}}\left(\boldsymbol{r}\right)=d_{i_{\mathrm{C}}}^2-l_{i_{\mathrm{C}}}^2=\boldsymbol{r}_{\mathrm{g},i_{\mathrm{C}}}{}^2-l_{i_{\mathrm{C}}}^2=0 \tag{42}$$

is satisfied at any time. Note, that the above equation is set up in terms of the squared lengths, since this eliminates the square root appearing in equation 41 thus making differentiation of the constraint equation more comfortable.

However, if the constraint is not satisfied as a result from errors originating from its numerical integration, there is a difference in the actual distance of the nodes and the specified length $l_{i_{\mathrm{C}}}$. It is expressed by the introduction of the term

$$\Delta l_{i_{\mathrm{C}}}=d_{i_{\mathrm{C}}}-l_{i_{\mathrm{C}}}\,, \tag{43}$$

which, unlike the actual constraint equation, is the difference of the actual lengths – not their squared values. To correct this violation by projection, both nodes are moved along the direction of the constraint, thus that Δl_{i_C} becomes 0. For this purpose, a projection-vector $\boldsymbol{r}_{\mathrm{pr},i_C}$ with the length of Δl_{i_C} is created in the direction of the constraint:

$$\boldsymbol{r}_{\mathrm{pr},i_C} = \boldsymbol{u}_{\mathrm{g},i_C} \Delta l_{i_C} \,. \tag{44}$$

Finally, $\boldsymbol{r}_{\mathrm{pr},i_C}$ is distributed to the two nodes by splitting it into two vectors of opposite direction. Here, the distribution of the projection-vector is weighed by the interchanged masses of the nodes

$$m_{1,i_C} \equiv m_{\mathrm{Sys},k1(i_C)} \;,\; m_{2,i_C} \equiv m_{\mathrm{Sys},k2(i_C)} \tag{45}$$

so that the node with the higher mass is shifted less due to it's higher inertia. The two vectors, by which the nodes are to by shifted, are then obtained as

$$\boldsymbol{r}_{\mathrm{pr1},i_C} = \boldsymbol{u}_{\mathrm{g},i_C} \frac{m_{2,i_C}}{m_{1,i_C} + m_{2,i_C}} \Delta l_{i_C} \tag{46}$$

and

$$\boldsymbol{r}_{\mathrm{pr2},i_C} = -\boldsymbol{u}_{\mathrm{g},i_C} \frac{m_{1,i_C}}{m_{1,i_C} + m_{2,i_C}} \Delta l_{i_C} \,. \tag{47}$$

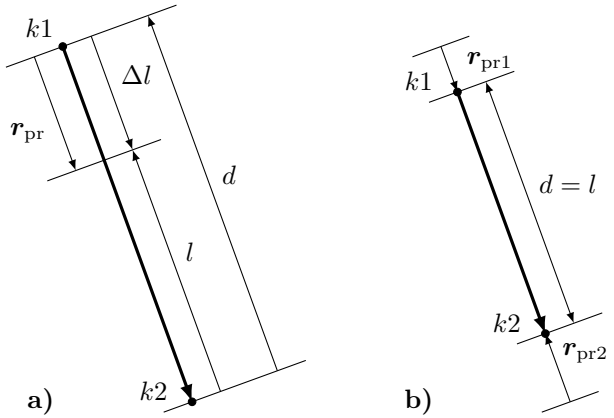

Figure 9 Projection of the positions. **a)** Constraint prior to projection **b)** Constraint after projection

Finally, the projection algorithm is implemented as an iteration over all constraints and repeated n_{Iter} times.

Relaxation Algorithm

FOR $i = 1$ **TO** $\ldots n_{\mathrm{Iter}}$ **DO**

FOREACH constraint g_{i_C} **DO**

$$\boldsymbol{r}^{i+1}_{k_{1/2}(i_C)} = \boldsymbol{r}^{i}_{k_{1/2}(i_C)} + \alpha \, \boldsymbol{r}_{\mathrm{pr1/2},i_C}$$

END FOREACH

END FOR

Here, α is introduced as weight factor in order to stabilise the solution. α is typically set to 0.5 but can be increased in order to increase the rate of convergence. Note that within one iteration step the positions $\boldsymbol{r}^{i}_{k_{1/2}[i_C]}$ might occur in several concurring constraints. In that case, the position already changed by the preceding projection of another constraint is used as a basis for the projection of the current constraint.

Reconstrcuted Reaction Forces - Type I (Velocity-Verlet)

- **Predictor step** *Velocity verlet* integration (refer to [6]) with subsequent relaxation and reaction force reconstruction:

$${}^{\mathrm{P}}\hat{\boldsymbol{r}}^{*}_{k+1} = \frac{1}{2}\hat{\boldsymbol{M}}^{-1}\left(\hat{\boldsymbol{f}}^{\mathrm{e}}_{k} + \cancel{\hat{\boldsymbol{f}}^{\mathrm{r}}_{k}}\right)\Delta t^2 + \dot{\hat{\boldsymbol{r}}}_{k}\Delta t + \hat{\boldsymbol{r}}_{k} \tag{48}$$

$${}^{\mathrm{P}}\hat{\boldsymbol{r}}_{k+1} = \mathrm{relax}({}^{\mathrm{P}}\hat{\boldsymbol{r}}^{*}_{k+1}) \tag{49}$$

- **Update velocities** Approximation as applied in [2], which according to the author might be inaccurate in certain situations, but does increase stability.

$${}^{\mathrm{P}}\hat{\boldsymbol{v}}_{k+1} = \left({}^{\mathrm{P}}\hat{\boldsymbol{r}}_{k+1} - \hat{\boldsymbol{r}}_{k}\right)/\Delta t \,. \tag{50}$$

- **Reconstruct Reaction forces**

$$\frac{\hat{\boldsymbol{f}}^{\mathrm{r}}_{k} + {}^{\mathrm{P}}\hat{\boldsymbol{f}}^{\mathrm{r}}_{k+1}}{2} \approx \frac{2\hat{\boldsymbol{M}}\left({}^{\mathrm{P}}\hat{\boldsymbol{r}}_{k+1} - \dot{\hat{\boldsymbol{r}}}_{k}\Delta t - \hat{\boldsymbol{r}}_{k}\right)}{\Delta t^2} - \hat{\boldsymbol{f}}^{\mathrm{e}}_{k} \,. \tag{51}$$

- **Trapezoidal rule** Integration based on forces due to predicted position ${}^{\mathrm{P}}\hat{\boldsymbol{r}}_{k+1}$. Includes further relaxation.

$$\begin{aligned}\hat{\boldsymbol{r}}^{*}_{k+1} &= \frac{1}{2}\hat{\boldsymbol{M}}^{-1}\left(\frac{\hat{\boldsymbol{f}}^{\mathrm{e}}_{k}+{}^{\mathrm{P}}\hat{\boldsymbol{f}}^{\mathrm{e}}_{k+1}}{2}+\frac{\hat{\boldsymbol{f}}^{\mathrm{r}}_{k}+{}^{\mathrm{P}}\hat{\boldsymbol{f}}^{\mathrm{r}}_{k+1}}{2}\right)\Delta t^{2}+\dot{\hat{\boldsymbol{r}}}_{k}\Delta t+\hat{\boldsymbol{r}}_{k} \\ &= \frac{1}{4}\hat{\boldsymbol{M}}^{-1}\left({}^{\mathrm{P}}\hat{\boldsymbol{f}}^{\mathrm{e}}_{k+1}-\hat{\boldsymbol{f}}^{\mathrm{e}}_{k}\right)\Delta t^{2}+{}^{\mathrm{P}}\hat{\boldsymbol{r}}_{k+1}\end{aligned} \tag{52}$$

$$\hat{\boldsymbol{r}}_{k+1} = \mathrm{relax}(\hat{\boldsymbol{r}}^{*}_{k+1})\,. \tag{53}$$

$$\hat{\boldsymbol{v}}_{k+1} = \left(\hat{\boldsymbol{r}}_{k+1}-\hat{\boldsymbol{r}}_{k}\right)/\Delta t\,. \tag{54}$$

Reconstrcuted Reaction Forces - Type II (Velocity-Verlet)

- **Predictor step** *Velocity verlet* integration with subsequent relaxation and reaction force reconstruction. Stabilisation of predictor step by partial inclusion of previous reaction forces:

$${}^{\mathrm{P}}\hat{\boldsymbol{r}}^{*}_{k+1} = \frac{1}{2}\hat{\boldsymbol{M}}^{-1}\left(\hat{\boldsymbol{f}}^{\mathrm{e}}_{k}+\alpha_{\mathrm{reac}}\frac{\hat{\boldsymbol{f}}^{\mathrm{r}}_{k-1}+\hat{\boldsymbol{f}}^{\mathrm{r}}_{k}}{2}\right)\Delta t^{2}+\dot{\hat{\boldsymbol{r}}}_{k}\Delta t+\hat{\boldsymbol{r}}_{k} \tag{55}$$

$${}^{\mathrm{P}}\hat{\boldsymbol{r}}_{k+1} = \mathrm{relax}({}^{\mathrm{P}}\hat{\boldsymbol{r}}^{*}_{k+1}) \tag{56}$$

$${}^{\mathrm{P}}\hat{\boldsymbol{v}}_{k+1} = \left({}^{\mathrm{P}}\hat{\boldsymbol{r}}_{k+1}-\hat{\boldsymbol{r}}_{k}\right)/\Delta t\,. \tag{57}$$

- **Trapezoidal rule**

$$\hat{\boldsymbol{r}}^{*}_{k+1} = \frac{1}{4}\hat{\boldsymbol{M}}^{-1}\left({}^{\mathrm{P}}\hat{\boldsymbol{f}}^{\mathrm{e}}_{k+1}-\hat{\boldsymbol{f}}^{\mathrm{e}}_{k}\right)\Delta t^{2}+{}^{\mathrm{P}}\hat{\boldsymbol{r}}_{k+1} \tag{58}$$

$$\hat{\boldsymbol{r}}_{k+1} = \mathrm{relax}(\hat{\boldsymbol{r}}^{*}_{k+1})\,. \tag{59}$$

$$\hat{\boldsymbol{v}}_{k+1} = \left(\hat{\boldsymbol{r}}_{k+1}-\hat{\boldsymbol{r}}_{k}\right)/\Delta t\,. \tag{60}$$

$$\frac{\hat{\boldsymbol{f}}^{\mathrm{r}}_{k}+\hat{\boldsymbol{f}}^{\mathrm{r}}_{k+1}}{2} \approx \frac{2\hat{\boldsymbol{M}}\left(\hat{\boldsymbol{r}}_{k+1}-\dot{\hat{\boldsymbol{r}}}_{k}\Delta t-\hat{\boldsymbol{r}}_{k}\right)}{\Delta t^{2}}-\frac{\hat{\boldsymbol{f}}^{\mathrm{e}}_{k}+{}^{\mathrm{P}}\hat{\boldsymbol{f}}^{\mathrm{e}}_{k}}{2}\,. \tag{61}$$

Internal reaction forces

The calculation of the internal reaction forces is based on implicit constraint equations and the method of LAGRANGE-multipliers as described in e.g. [5]. In order to derive the general algorithm, the four node exemplary structure shown in figure 10 is analysed first.

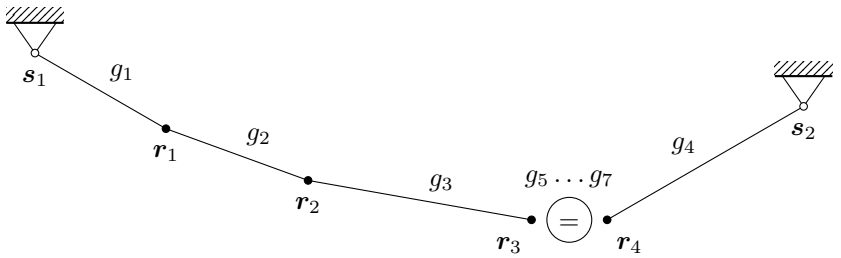

Figure 10 Exemplary system with a secondary constraint between $\boldsymbol{r}_3$ and $\boldsymbol{r}_4$

Using equation 42, the implicit constraint equations of this system can be specified as

$$\boldsymbol{g}(\boldsymbol{r}) = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{bmatrix} = \left[\begin{array}{c} (r_{1x}-s_{1x})^2+(r_{1y}-s_{1y})^2+(r_{1z}-s_{1z})^2-l_1^2 \\ (r_{2x}-r_{1x})^2+(r_{2y}-r_{1y})^2+(r_{2z}-r_{1z})^2-l_2^2 \\ (r_{3x}-r_{2x})^2+(r_{3y}-r_{2y})^2+(r_{3z}-r_{2z})^2-l_3^2 \\ (r_{4x}-s_{2x})^2+(r_{4y}-s_{2y})^2+(r_{4z}-s_{2z})^2-l_4^2 \\ \hline r_{4x}-r_{3x} \\ r_{4y}-r_{3y} \\ r_{4z}-r_{3z} \end{array}\right] = \overline{\mathbf{0}}_7\,, \tag{62}$$

which can be written in more compact form as

$$\boldsymbol{g}(\boldsymbol{r}) = \left[\begin{array}{c} (\boldsymbol{r}_1-\boldsymbol{s}_1)^2-l_1^2 \\ (\boldsymbol{r}_2-\boldsymbol{r}_1)^2-l_2^2 \\ (\boldsymbol{r}_3-\boldsymbol{r}_2)^2-l_3^2 \\ (\boldsymbol{r}_4-\boldsymbol{s}_2)^2-l_4^2 \\ \hline \boldsymbol{r}_4-\boldsymbol{r}_3 \end{array}\right] = \overline{\mathbf{0}}_7\,. \tag{63}$$

Partial differentiation with respect to $\boldsymbol{r}_i$ leads to the JACOBIAN-matrices $\boldsymbol{G}_i$ of the implicit constraints. Thus, for instance, $\boldsymbol{G}_2$ can be retrieved as

$$\boldsymbol{G}_2 = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}_2} = 2\left[\begin{array}{ccc} 0 & 0 & 0 \\ (r_{2x}-r_{1x}) & (r_{2y}-r_{1y}) & (r_{2z}-r_{1z}) \\ -(r_{3x}-r_{2x}) & (r_{3y}-r_{2y}) & (r_{3z}-r_{2z}) \\ 0 & 0 & 0 \\ \hline \overline{\mathbf{0}}_3 & \overline{\mathbf{0}}_3 & \overline{\mathbf{0}}_3 \end{array}\right] = 2\left[\begin{array}{c} 0 \\ (\boldsymbol{r}_2-\boldsymbol{r}_1)^{\mathrm{T}} \\ -(\boldsymbol{r}_3-\boldsymbol{r}_2)^{\mathrm{T}} \\ 0 \\ \hline \overline{\mathbf{0}}_3 \end{array}\right]. \tag{64}$$

Gathering the JACOBIAN matrices of all constraints, the system's JACOBIAN can finally bes specified as

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{G}_1 & \dots & \boldsymbol{G}_4 \end{bmatrix} = 2\left[\begin{array}{cccc} (\boldsymbol{r}_1-\boldsymbol{s}_1)^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\boldsymbol{r}_2-\boldsymbol{r}_1)^{\mathrm{T}} & (\boldsymbol{r}_2-\boldsymbol{r}_1)^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -(\boldsymbol{r}_3-\boldsymbol{r}_2)^{\mathrm{T}} & (\boldsymbol{r}_3-\boldsymbol{r}_2)^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\boldsymbol{r}_4-\boldsymbol{s}_2)^{\mathrm{T}} \\ \hline \mathbf{0} & \mathbf{0} & -\tfrac{1}{2}\boldsymbol{I}_3 & \tfrac{1}{2}\boldsymbol{I}_3 \end{array}\right]. \tag{65}$$

According to the LAGRANGE-D'ALEMBERT principle, the reaction forces acting on mass point i may only act in the direction of the implicit constraints, which are defined by the row vectors of $\boldsymbol{G}$. Accordingly, the reaction forces can be expressed as a linear combination of $\boldsymbol{G}$ and a set of n_{C} so far unknown LAGRANGE-multipliers $\boldsymbol{\lambda}$ as

$$\boldsymbol{f}^{\mathrm{r}} = \begin{bmatrix} \boldsymbol{f}_1^{\mathrm{r}} \\ \boldsymbol{f}_2^{\mathrm{r}} \\ \boldsymbol{f}_3^{\mathrm{r}} \\ \boldsymbol{f}_4^{\mathrm{r}} \end{bmatrix} = 2\left[\begin{array}{cccc|c} (\boldsymbol{r}_1-\boldsymbol{s}_1) & -(\boldsymbol{r}_2-\boldsymbol{r}_1) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\boldsymbol{r}_2-\boldsymbol{r}_1) & -(\boldsymbol{r}_3-\boldsymbol{r}_2) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\boldsymbol{r}_3-\boldsymbol{r}_2) & \mathbf{0} & -\tfrac{1}{2}\boldsymbol{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\boldsymbol{r}_4-\boldsymbol{s}_2) & \tfrac{1}{2}\boldsymbol{I}_3 \end{array}\right] \left[\begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \hline \lambda_5 \end{array}\right]. \tag{66}$$

Given the total reaction forces $\boldsymbol{f}_i^{\mathrm{r}}$ according to **??**, this linear system of equations can now be solved for $\lambda_1 \dots \lambda_5$.

When the LAGRANGE-multipliers are known, the forces associated with each single constraint can be determined. As to be seen from the example given above, each constraint equation generates internal reaction forces acting on two nodes. This is in accordance with NEWTON's third law. Thus, for instance, constraint g_2 restricts the motion of nodes $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. As to be seen from equation (66), this causes the same internal reaction

$$\boldsymbol{f}_2^{\mathrm{b}} = 2\left(\boldsymbol{r}_2 - \boldsymbol{r}_1\right)\lambda_2 \tag{67}$$

force acting on both nodes with an inverted sign. Applying these considerations to all nodes, the internal reaction forces can be obtained from

$$\begin{bmatrix} \boldsymbol{f}_1^{\mathrm{r}} \\ \boldsymbol{f}_2^{\mathrm{r}} \\ \boldsymbol{f}_3^{\mathrm{r}} \\ \boldsymbol{f}_4^{\mathrm{r}} \end{bmatrix} = 2 \begin{bmatrix} \left(\boldsymbol{r}_1 - \boldsymbol{s}_1\right)\lambda_1 - \left(\boldsymbol{r}_2 - \boldsymbol{r}_1\right)\lambda_2 \\ \left(\boldsymbol{r}_2 - \boldsymbol{r}_1\right)\lambda_2 - \left(\boldsymbol{r}_3 - \boldsymbol{r}_2\right)\lambda_3 \\ \left(\boldsymbol{r}_3 - \boldsymbol{r}_2\right)\lambda_3 \\ \left(\boldsymbol{r}_4 - \boldsymbol{s}_2\right)\lambda_4 \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ -\boldsymbol{I}_3 \\ \boldsymbol{I}_3 \end{bmatrix} \begin{bmatrix} \lambda_5 \\ \lambda_6 \\ \lambda_7 \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1^{\mathrm{b}} - \boldsymbol{f}_2^{\mathrm{b}} \\ \boldsymbol{f}_2^{\mathrm{b}} - \boldsymbol{f}_3^{\mathrm{b}} \\ \boldsymbol{f}_3^{\mathrm{b}} \\ \boldsymbol{f}_4^{\mathrm{b}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ -\boldsymbol{I}_3 \\ \boldsymbol{I}_3 \end{bmatrix} \begin{bmatrix} \lambda_5 \\ \lambda_6 \\ \lambda_7 \end{bmatrix} . \tag{68}$$

In order to conclude a general relationship from these derivations, consider a general constant distant constraint g_i between two nodes $k1\,(i_{\mathrm{C}})$ and $k2\,(i_{\mathrm{C}})$

$$g_i = \left(\boldsymbol{r}_{k2(i_{\mathrm{C}})} - \boldsymbol{r}_{k1(i_{\mathrm{C}})}\right)^2 - l_i^2 \,. \tag{69}$$

Once the LAGRANGE-multipliers are known in accordance to the procedure illustrated by the above example, the internal reaction force associated with g_i can be determined as

$$\boldsymbol{f}_i^{\mathrm{b}} = 2\left(\boldsymbol{r}_{k2(i_{\mathrm{C}})} - \boldsymbol{r}_{k1(i_{\mathrm{C}})}\right)\lambda_i \,. \tag{70}$$

References

[1] Jan Bender, Matthias Müller, Miguel A. Otaduy, and Matthias Teschner. Position-based methods for the simulation of solid objects in computer graphics. In *Eurographics 2013-State of the Art Reports*, pages 1–22. The Eurographics Association, 2012.

[2] Thomas Jakobsen. Advanced character physics. In *Game Developers Conference*, pages 383–401, 2001.

[3] Christoph Otto, Thomas Miethe, and Mathias Paschen. Extension of the reconstructed reaction forces formulation and application to an aquaculture cage. In *Contributions on the Theory of Fishing Gears and Related Marine Systems*, pages 183–192, 2015.

[4] Christoph Otto and Mathias Paschen. A new approach to the simulation of ideally flexible cloth-like structures. In *Contributions on the Theory of Fishing Gears and Related Marine Systems*, pages 63–72, 2013.

[5] Ahmed A Shabana. *Dynamics of multibody systems*. Cambridge University Press, 2005.

[6] William C. Swope, Hans C. Andersen, Peter H. Berens, and Kent R. Wilson. A computer simulation method for the calculation of equilibrium constants for the formation of physical clusters of molecules: Application to small water clusters. *The Journal of Chemical Physics*, 76:637, 1982.

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