

Syntactic presentations for glued toposes and for crystalline toposes

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*Im Gedenken an meinen Papa,
der mich auf seine Schultern hob.*

Abstract

We regard a geometric theory classified by a topos as a syntactic presentation for the topos and develop tools for finding such presentations. Extensions (or expansions) of geometric theories, which can not only add axioms but also symbols and sorts, are treated as objects in their own right, to be able to build up complex theories from parts. The role of equivalence extensions, which leave the theory the same up to Morita equivalence, is investigated.

Motivated by the question what the big Zariski topos of a non-affine scheme classifies, we show how to construct a syntactic presentation for a topos if syntactic presentations for a covering family of open subtoposes are given. For this, we introduce a transformation of theory extensions such that when the result, dubbed a conditional extension, is added to a theory, it requires part of the data a model is made of only under some condition given in the form of a closed geometric formula. We also give a general definition for systems of interdependent theory extensions, to be able to talk about compatible syntactic presentations not only for the open subtoposes in a given cover but also for their finite intersections.

An important concept for finding classified theories of toposes in concrete situations is that of theories of presheaf type. We develop several techniques for extending a theory while preserving the presheaf type property, and give a list of examples of simple extensions which can destroy it.

Finally, we determine a syntactic presentation of the big crystalline topos of a scheme. In the case of an affine scheme, this is accomplished by showing that the biggest part of the classified theory is of presheaf type and transforming the site defining the crystalline topos into the canonical presheaf site for this theory, while the remaining axioms induce the Zariski topology. Then we can apply our results on gluing classifying toposes to obtain a classified theory even in the non-affine case.

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1 Introduction

This thesis elaborates in various ways on the theme in topos theory that a Grothendieck topos can be viewed as *the essence of a geometric theory*. Formally, one says that a Grothendieck topos \mathcal{E} *classifies* the geometric theory \mathbb{T} if the models of \mathbb{T} in any Grothendieck topos \mathcal{E}' correspond to the geometric morphisms from \mathcal{E}' to \mathcal{E} . Since the topos \mathcal{E} is then uniquely determined by the theory \mathbb{T} , we can take the dual standpoint that the theory \mathbb{T} is a *presentation* for the topos \mathcal{E} . (All theories that we will meet will be geometric theories, and the term topos will always mean Grothendieck topos.)

One reason why we might seek such a presentation for a given topos is that it can be much more concise and, we would argue, even more intuitive than a definition of the same topos by a site. The prime example for this is the big Zariski topos $\mathcal{E} = (\text{Spec } \mathbb{Z})_{\text{Zar}}$ of the affine scheme $\text{Spec } \mathbb{Z}$. A site of definition for \mathcal{E} is given by the opposite category of the category of all finitely presentable rings, equipped with a certain Grothendieck topology called the Zariski topology, which involves localizations A_a of a ring A at an element $a \in A$ and the condition that some elements generate the unit ideal $(1) = A$. On the other hand, one can also define \mathcal{E} as the classifying topos of the theory of local rings. Here, a full definition consists simply in writing down the usual algebraic operations and axioms defining a (commutative, unitary) ring and the extra axiom that the ring be local, stated in the elementary form that if a sum of elements is invertible, then one of these elements is invertible too. Of course, to get an actual Grothendieck topos out of this, one needs the whole machinery of classifying toposes, but the presentation itself is quite short and very approachable. To a certain degree, it is even possible to judge manipulations of the syntactic presentation correctly, based on nothing but intuition from elementary algebra. For example, the classifying topos stays the same if we add to the list of axioms a redundant one like $(xy)z = (zy)x$, but not if we add an axiom like $x = -x$.

Another reason is simply that a classified geometric theory for a topos \mathcal{E} is a description of the representable functor $\text{Hom}(-, \mathcal{E})$, that is, it is a definition for \mathcal{E} by a universal property. We would like to stress that since Grothendieck toposes form a 2-category, the representable functor $\text{Hom}(-, \mathcal{E})$ is in fact a pseudofunctor, from the 2-category of Grothendieck toposes to the 2-category of categories. Such a pseudofunctor comprises a huge amount of data, and it is notoriously difficult to keep track of the coherence conditions that this data must satisfy. In contrast, it is simple to check whether a geometric theory is well-defined, and while it can in general contain arbitrarily big sets (of relation and function symbols, say) as well, the examples showing up in practice are often more or less finitary.

It should be mentioned here that syntactic presentations of toposes do always exist, and there is a clear procedure for constructing a classified geometric theory out of a given site presentation of a topos. But a presentation constructed in this way will of course generally not tell us anything more about the topos than the site itself does. Whenever we speak of searching for syntactic presentations, we intend to find a *concise* presentation, or one that is interesting for some other reason.

The first of our two main goals, which will occupy us in Sections 2 and 3, will be

to give a construction on the level of geometric theories for an operation which is very natural when viewing toposes as generalized topological spaces, namely the operation of gluing toposes along open subtoposes. More precisely, our setup will be that \mathcal{E} is a topos covered by open subtoposes $\mathcal{E}_i = \mathcal{E}_{o(U_i)}$, and syntactic presentations for the \mathcal{E}_i are given. Then we ask how to construct a syntactic presentation for \mathcal{E} , and what additional data might be needed for this. The appropriate gluing data will consist, unsurprisingly, of syntactic presentations of the intersections of the \mathcal{E}_i , but not given independently of those for the \mathcal{E}_i , but rather compatible with them, or, really, *extending* them. Here, the notion of extensions of geometric theories will be crucial, which will therefore be investigated first. The formula we give for the theory classified by \mathcal{E} (see Theorem 3.5.4) will then be quite elegant, it simply adds up all the given theory extensions, after transforming them into theory extensions for “partial models” over the respective open subtoposes. This gluing technique is then applied to deduce a syntactic presentation for the big Zariski topos of a non-affine scheme (see Theorem 3.7.6) from the well-known result in the affine case.

Our second objective, in Sections 4 and 5, is to give syntactic presentations for another family of toposes from algebraic geometry, namely the crystalline toposes of schemes. These toposes were introduced around 1970 to study crystalline cohomology, a tool for extracting geometric information from schemes, similar to de Rham cohomology, but specifically adapted to schemes over ground fields of positive characteristic. While more and more classified theories for other toposes from algebraic geometry were found over the years, a syntactic presentation of the crystalline topos was up to now missing. The construction of a crystalline topos depends in fact not on a single scheme, as for the Zariski topos, but on two schemes with some additional structure. This is reflected in the more involved classified theory we give (see Theorem 5.7.4), but it is still very much related to the theory of local rings, and the universal model living in the crystalline topos consists precisely of its structure sheaf and some additional data associated to it. The case of affine base schemes is treated first, and relies heavily on techniques for recognizing theories of presheaf type which we develop for this purpose. It is then simply another application of the gluing theorem to generalize to the non-affine case (see Theorem 5.8.3), although some extra care is needed in constructing an open cover and a system of syntactic presentations for it.

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2 Extensions of geometric theories

2.1 Background on geometric theories

For a full definition of geometric theories, we refer to [5, Chapter D1.1]. But we want to mention that a geometric theory \mathbb{T} can be thought of as consisting of three “layers”, first come the *sorts*, then the *relation* and *function symbols*, and finally the *axioms*. The first two layers are often called the *signature* of the theory. What is allowed in each layer depends on the data in the previous layers: The set of sorts of \mathbb{T} is just a set without any additional structure. The relation and function symbols have their own signatures, which in this case just means a list of sorts, and which we denote

$$R \subseteq A_1 \times \cdots \times A_n \quad \text{and} \quad f : A_1 \times \cdots \times A_n \rightarrow B,$$

where A_1, \dots, A_n and B are sorts of \mathbb{T} . And the axioms are sequents of the form

$$\phi \vdash_{x_1 : A_1, \dots, x_n : A_n} \psi,$$

to be read as “ ϕ implies ψ in the context $x_1 : A_1, \dots, x_n : A_n$ ”, where the geometric formulas ϕ and ψ can use the relation and function symbols of \mathbb{T} . The context will sometimes be abbreviated $\vec{x} : \vec{A}$.

A relation symbol with the empty signature, $n = 0$, which we might denote $R \subseteq 1$, is called a *proposition symbol*, and we will rather use the letter p for it. Similarly, a function symbol with empty domain, $n = 0$, is called a *constant symbol*, and instead of $f : 1 \rightarrow B$, we simply denote it as $f : B$, or rather $c : B$. A geometric theory is *propositional* if it has no sorts, and therefore also no function symbols and no relation symbols except proposition symbols.

A *model* of a geometric theory \mathbb{T} in a topos \mathcal{E} consists of objects

$$\llbracket A \rrbracket_M \in \mathcal{E}$$

for all sorts A of \mathbb{T} , subobjects

$$\llbracket R \rrbracket_M \subseteq \llbracket A_1 \rrbracket_M \times \cdots \times \llbracket A_n \rrbracket_M$$

for all relation symbols R of \mathbb{T} and morphisms

$$\llbracket f \rrbracket_M : \llbracket A_1 \rrbracket_M \times \cdots \times \llbracket A_n \rrbracket_M \rightarrow \llbracket B \rrbracket_M$$

for all function symbols f of \mathbb{T} , such that the axioms of \mathbb{T} are fulfilled. Given a model M , we can not only *interpret* individual sorts and symbols in M , but also any geometric formula ϕ of \mathbb{T} in a context $x_1 : A_1, \dots, x_n : A_n$, yielding a subobject

$$\llbracket \phi \rrbracket_M \subseteq \llbracket A_1 \rrbracket_M \times \cdots \times \llbracket A_n \rrbracket_M.$$

The requirement that an axiom

$$\phi \vdash_{\vec{x} : \vec{A}} \psi$$

is fulfilled in M means that there is an inclusion of subobjects

$$\llbracket \phi \rrbracket_M \leq \llbracket \psi \rrbracket_M.$$

There is also a notion of *morphism* between models of the same theory in the same topos, and the resulting category of \mathbb{T} -models in \mathcal{E} will be denoted

$$\mathbb{T}\text{-mod}(\mathcal{E}).$$

Furthermore, the requirement that the axioms of \mathbb{T} are geometric sequents ensures that pulling back the individual parts of a model $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ along a geometric morphism $f : \mathcal{E}' \rightarrow \mathcal{E}$ yields a model of \mathbb{T} in \mathcal{E}' . For a fixed model $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ and varying f , this constitutes a functor

$$\text{Geom}(\mathcal{E}', \mathcal{E}) \rightarrow \mathbb{T}\text{-mod}(\mathcal{E}'),$$

and the model M is called a *universal model* of \mathbb{T} if this functor is an equivalence of categories for all (Grothendieck) toposes \mathcal{E}' . The topos \mathcal{E} is then called a *classifying topos* for the geometric theory \mathbb{T} , and we will call the pair (\mathbb{T}, M) a *syntactic presentation* of the topos \mathcal{E} .

It is a theorem that every (Grothendieck) topos classifies some geometric theory and every geometric theory admits a classifying topos, that is, a universal model in some topos. The classifying topos of a theory is also unique up to equivalence, which justifies writing

$$\text{Set}[\mathbb{T}]$$

for a classifying topos of a theory \mathbb{T} . But it is not at all true that the classified theory of a topos is unique. Instead, two theories admitting universal models in the same topos are called *Morita equivalent* theories.

There is of course also a notion of provability for geometric theories, which we will not define here. A geometric sequent which is provable in a geometric theory \mathbb{T} is fulfilled in any model of \mathbb{T} in any topos, so it could just as well be added as an axiom of \mathbb{T} . Two theories \mathbb{T}_1 and \mathbb{T}_2 over the same signature (same sorts and symbols) are called *syntactically equivalent* if every axiom of \mathbb{T}_1 is provable in \mathbb{T}_2 and vice versa. Note that this is a much stronger condition on \mathbb{T}_1 and \mathbb{T}_2 than being Morita equivalent, since Morita equivalent theories can have different signatures. We will simply write syntactic equivalence as equality,

$$\mathbb{T}_1 = \mathbb{T}_2.$$

The universal model of a theory \mathbb{T} is unique in the sense that for any two universal models in toposes \mathcal{E}_1 and \mathcal{E}_2 , there is an equivalence $\mathcal{E}_1 \simeq \mathcal{E}_2$ sending one to the other. With respect to provability, it has the following strong property. A geometric sequent of a theory \mathbb{T} is fulfilled in the universal model of \mathbb{T} if and only if it is provable in \mathbb{T} .

Finally, we would like to make the point that when manipulating geometric theories, one has to think intuitionistically. This is not the case when the matter is only about axioms; it is a well-known theorem that any geometric sequent which is provable from the axioms of a geometric theory using full classical first-order logic is also provable from

these axioms in geometric logic. But if we are also interested in adding sorts and symbols, the intuitionistic nature shows clearly. For example, the theory consisting of a single proposition symbol p has two models (up to isomorphism) in Set , as the interpretation of p can be either true or false. The same is true for the theory with two proposition symbols p and q and the axioms

$$\top \vdash_{\square} p \vee q \quad \text{and} \quad p \wedge q \vdash_{\square} \perp.$$

But these are two completely different theories, as a model of the first theory in a topos \mathcal{E} is just an open subtopos of \mathcal{E} , while a model of the second theory is a decomposition of \mathcal{E} into two subtoposes which are both open and closed. And even the categories of models in Set are not equivalent, since proposition symbols are allowed to become true but not to become false under model homomorphisms.

2.2 Theory extensions as presentations of geometric morphisms

Definition 2.2.1. A (geometric) *extension* \mathbb{E} of a geometric theory \mathbb{T} consists of a set $\mathbb{E}\text{-Sort}$ of sorts, sets $\mathbb{E}\text{-Rel}$ and $\mathbb{E}\text{-Fun}$ of relation and function symbols over the sorts $\mathbb{T}\text{-Sort} \sqcup \mathbb{E}\text{-Sort}$ and a set of geometric axioms over the sorts $\mathbb{T}\text{-Sort} \sqcup \mathbb{E}\text{-Sort}$ and the symbols $\mathbb{T}\text{-Rel} \sqcup \mathbb{E}\text{-Rel}$ and $\mathbb{T}\text{-Fun} \sqcup \mathbb{E}\text{-Fun}$. We denote $\mathbb{T} + \mathbb{E}$ the theory obtained by adding these sorts, symbols and axioms to \mathbb{T} . The extension \mathbb{E} is *localic* if $\mathbb{E}\text{-Sort} = \emptyset$; it is a *quotient extension* if additionally $\mathbb{E}\text{-Rel} = \emptyset$ and $\mathbb{E}\text{-Fun} = \emptyset$.

If \mathbb{E} is an extension of \mathbb{T} , we have a forgetful functor

$$U_{\mathbb{E}} : (\mathbb{T} + \mathbb{E})\text{-mod}(\mathcal{E}) \rightarrow \mathbb{T}\text{-mod}(\mathcal{E})$$

for every Grothendieck topos \mathcal{E} . Note that this functor is an isofibration. Also, after fixing classifying toposes $\text{Set}[\mathbb{T}]$ and $\text{Set}[\mathbb{T} + \mathbb{E}]$, the \mathbb{T} -model part of the universal $(\mathbb{T} + \mathbb{E})$ -model induces a canonical geometric morphism

$$\pi_{\mathbb{E}} : \text{Set}[\mathbb{T} + \mathbb{E}] \rightarrow \text{Set}[\mathbb{T}],$$

which in turn acts on generalized points by the functors $U_{\mathbb{E}}$ (up to natural isomorphism). This is the geometric morphism *presented* by the extension \mathbb{E} .

The following theorem says that every geometric morphism can be presented in this way, thus generalizing the result that every Grothendieck topos classifies a geometric theory to the relative situation over some base topos $\text{Set}[\mathbb{T}]$ with an already chosen syntactic presentation.

Theorem 2.2.2 ([4, Theorem 7.1.5]). *Let a geometric morphism*

$$p : \mathcal{E} \rightarrow \text{Set}[\mathbb{T}]$$

to the classifying topos of a geometric theory \mathbb{T} be given. Then p is, up to isomorphism, of the form $\pi_{\mathbb{E}} : \text{Set}[\mathbb{T} + \mathbb{E}] \rightarrow \text{Set}[\mathbb{T}]$ for some extension \mathbb{E} of \mathbb{T} . If p is localic (respectively an embedding), then we can take \mathbb{E} to be a localic extension (respectively a quotient extension).

Proof. See [4, Theorem 7.1.5]. The case of an embedding, which is not mentioned there, is instead part of the duality between subtoposes and quotient theories [4, Theorem 3.2.5]. \square

Given two extensions \mathbb{E}_1 and \mathbb{E}_2 of a theory \mathbb{T} , it is clear that we can also regard \mathbb{E}_2 as an extension of $\mathbb{T} + \mathbb{E}_1$ and vice versa. Then we have a strict pullback diagram of categories

$$\begin{array}{ccc} (\mathbb{T} + \mathbb{E}_1 + \mathbb{E}_2)\text{-mod}(\mathcal{E}) & \longrightarrow & (\mathbb{T} + \mathbb{E}_2)\text{-mod}(\mathcal{E}) \\ \downarrow & \lrcorner & \downarrow \\ (\mathbb{T} + \mathbb{E}_1)\text{-mod}(\mathcal{E}) & \longrightarrow & \mathbb{T}\text{-mod}(\mathcal{E}) \end{array}$$

for any topos \mathcal{E} , which is also a weak pullback, since the the forgetful functors are isofibrations. This means that $\text{Set}[\mathbb{T} + \mathbb{E}_1 + \mathbb{E}_2]$ is the pullback topos of $\text{Set}[\mathbb{T} + \mathbb{E}_1]$ and $\text{Set}[\mathbb{T} + \mathbb{E}_2]$ over $\text{Set}[\mathbb{T}]$. In particular, the product (as generalized spaces, not as categories) of two classifying toposes $\text{Set}[\mathbb{T}_1]$ and $\text{Set}[\mathbb{T}_2]$ classifies the theory $\mathbb{T}_1 + \mathbb{T}_2$.

In the same way in which we prefer to have a symbol for an extension \mathbb{E} instead of only for the extended theory $\mathbb{T}' = \mathbb{T} + \mathbb{E}$, we will also want to regard the data that is missing in a \mathbb{T} -model compared to a $(\mathbb{T} + \mathbb{E})$ -model as an object in its own right.

Definition 2.2.3. A *model extension* E of a model $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ along a theory extension \mathbb{E} of \mathbb{T} consists of interpretations for the sorts and symbols of \mathbb{E} that extend M to a model $M + E \in (\mathbb{T} + \mathbb{E})\text{-mod}(\mathcal{E})$. A *homomorphism* of model extensions of M is a family of maps, one for each sort of \mathbb{E} , that constitutes a $(\mathbb{T} + \mathbb{E})$ -model homomorphism when combined with the identity homomorphism of M . That is, the category of model extensions of M along \mathbb{E} is isomorphic to the strict preimage of M under $U_{\mathbb{E}}$; it will be denoted $\mathbb{E}\text{-mod}(\mathcal{E}, M)$ or simply $\mathbb{E}\text{-mod}(M)$.

Remark 2.2.4. Note that our terminology here is somewhat in conflict with the usage of for example “elementary extension” in set theory, which means a bigger model of the same theory.

Using the notion of model extensions, Theorem 2.2.2 can be formulated as follows. Given any model $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ of a geometric theory in some topos, there is always an extension \mathbb{E} of \mathbb{T} and a model extension E of M along \mathbb{E} such that the model $(M + E) \in (\mathbb{T} + \mathbb{E})\text{-mod}(\mathcal{E})$ is universal.

2.3 Equivalence extensions

Definition 2.3.1. An *equivalence extension* is an extension \mathbb{E} of \mathbb{T} such that the forgetful functor $U_{\mathbb{E}} : (\mathbb{T} + \mathbb{E})\text{-mod}(\mathcal{E}) \rightarrow \mathbb{T}\text{-mod}(\mathcal{E})$ is an equivalence of categories for every Grothendieck topos \mathcal{E} . Equivalently, the geometric morphism $\pi_{\mathbb{E}} : \text{Set}[\mathbb{T} + \mathbb{E}] \rightarrow \text{Set}[\mathbb{T}]$ presented by \mathbb{E} is an equivalence.

If \mathbb{E} is an equivalence extension of a theory \mathbb{T} , then it is also an equivalence extension of $\mathbb{T} + \mathbb{E}'$, for any other extension \mathbb{E}' of \mathbb{T} . This is clear from the (weak) pullback property of the category $(\mathbb{T} + \mathbb{E}_1 + \mathbb{E}_2)\text{-mod}(\mathcal{E})$.

Lemma 2.3.2. *An extension \mathbb{E} of a theory \mathbb{T} is an equivalence extension if and only if every model $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ in every Grothendieck topos \mathcal{E} admits exactly one model extension along \mathbb{E} up to isomorphism.*

Proof. The given condition means that $U_{\mathbb{E}} : (\mathbb{T} + \mathbb{E})\text{-mod}(\mathcal{E}) \rightarrow \mathbb{T}\text{-mod}(\mathcal{E})$ is bijective on isomorphism classes for every topos \mathcal{E} , which at first sight seems weaker than being an equivalence. But it means that the functors $\text{Geom}(\mathcal{E}, \text{Set}[\mathbb{T} + \mathbb{E}]) \rightarrow \text{Geom}(\mathcal{E}, \text{Set}[\mathbb{T}])$ induced by the geometric morphism $\pi_{\mathbb{E}} : \text{Set}[\mathbb{T} + \mathbb{E}] \rightarrow \text{Set}[\mathbb{T}]$ are bijective on isomorphism classes, in other words, $\pi_{\mathbb{E}}$ is an isomorphism in the 1-category of toposes and geometric morphisms up to isomorphism, which is the same as an equivalence of toposes. \square

Remark 2.3.3. Lemma 2.3.2 is a version of the slogan that if we know the models of a geometric theory, we also know the morphisms between them, see [5, below Lemma B4.2.3]. However, here it does not suffice to simply say that a morphism of \mathbb{T} -models in \mathcal{E} is the same as a \mathbb{T} -model in $\mathcal{E}^{\rightarrow}$, as $U_{\mathbb{E}}$ can be bijective on isomorphism classes for both \mathcal{E} and $\mathcal{E}^{\rightarrow}$ without being an equivalence for \mathcal{E} . For example, let $\mathbb{T} = \emptyset$, let \mathbb{E} be the theory of G -torsors for a group G and let $\mathcal{E} = \text{Set}$. Denote \underline{G} the one object groupoid associated to G . Then both of the functors

$$\begin{aligned} \underline{G} &\simeq (\mathbb{T} + \mathbb{E})\text{-mod}(\text{Set}) \rightarrow \mathbb{T}\text{-mod}(\text{Set}) \simeq \{*\}, \\ \underline{G} &\simeq \underline{G}^{\rightarrow} \simeq (\mathbb{T} + \mathbb{E})\text{-mod}(\text{Set}^{\rightarrow}) \rightarrow \mathbb{T}\text{-mod}(\text{Set}^{\rightarrow}) \simeq \{*\}^{\rightarrow} \simeq \{*\}, \end{aligned}$$

are bijective on isomorphism classes, but the first is not an equivalence.

Lemma 2.3.4. *Let $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ be a universal model, \mathbb{E} an extension of \mathbb{T} and E a model extension of M along \mathbb{E} . Then \mathbb{E} is an equivalence extension if and only if $M + E \in (\mathbb{T} + \mathbb{E})\text{-mod}(\mathcal{E})$ is again universal.*

Proof. This is immediate from the two-out-of-three property of equivalences of categories in the diagram

$$\begin{array}{ccc} & & (\mathbb{T} + \mathbb{E})\text{-mod}(\mathcal{E}') \\ & \nearrow^{-* (M+E)} & \downarrow U_{\mathbb{E}} \\ \text{Geom}(\mathcal{E}', \mathcal{E}) & & \mathbb{T}\text{-mod}(\mathcal{E}') \\ & \searrow_{-* M} & \end{array}$$

\square

Apart from syntactic equivalences, one way to produce an equivalence extension of a theory \mathbb{T} is to add new symbols for relations or functions that were already definable in \mathbb{T} . And in fact, all localic equivalence extensions are of this form up to syntactic equivalence, meaning that we have a simple syntactic characterization of equivalence extensions among the localic extensions.

Definition 2.3.5. Let \mathbb{T} be a geometric theory, let $R_i \subseteq \vec{A}^i$ and $f_j : \vec{B}^j \rightarrow C^j$ be families of new relation and function symbols (with signatures consisting of sorts of \mathbb{T})

and let $\{\vec{x}: \vec{A}^i . \phi_i(\vec{x})\}$ and $\{\vec{y}: \vec{B}^j, z: C^j . \psi_j(\vec{y}, z)\}$ be formulas of \mathbb{T} in corresponding contexts, where every ψ_j is provably functional. Then the *extension by definitions* for this data is the localic extension of \mathbb{T} consisting of the symbols R_i, f_j and the axioms

$$R_i(\vec{x}) \dashv\vdash_{\vec{x}: \vec{A}^i} \phi_i(\vec{x}) \quad \text{and} \quad f_j(\vec{y}) = z \dashv\vdash_{\vec{y}: \vec{B}^j, z: C^j} \psi_j(\vec{y}, z).$$

Remark 2.3.6. To show that a given localic extension \mathbb{E} of a theory \mathbb{T} is (syntactically equivalent to) an extension by definitions, one has to find formulas ϕ_i, ψ_j of \mathbb{T} , the latter provably functional, for the relation and function symbols of \mathbb{E} , such that the axioms in Definition 2.3.5 are provable in $\mathbb{T} + \mathbb{E}$, and furthermore, one has to check that all axioms of \mathbb{E} are already provable in \mathbb{T} if the symbols R_i, f_j are replaced by the formulas ϕ_i, ψ_j . However, this last check can be omitted if a model extension E along \mathbb{E} of a universal \mathbb{T} -model M is available, since then any \mathbb{T} -sequent which is true in $M + E$ is trivially also true in M and therefore provable in \mathbb{T} . For the same reason, it is then automatic that the ψ_j are provably functional. This will be relevant in applications of Corollary 3.6.1.

Proposition 2.3.7. *A localic extension is an equivalence extension if and only if it is (syntactically equivalent to) an extension by definitions.*

Proof. If \mathbb{E} is an extension by definitions of \mathbb{T} then any model $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ admits exactly one model extension E along \mathbb{E} , namely $\llbracket R_i \rrbracket_{M+E} := \llbracket \phi_i \rrbracket_M$ and $\llbracket f_j \rrbracket_{M+E}$ is the morphism with graph $\llbracket \psi_j \rrbracket_M$. So \mathbb{E} is an equivalence extension by Lemma 2.3.2.

On the other hand, if \mathbb{E} is an equivalence extension of \mathbb{T} , then consider a universal model $M \in \mathbb{T}\text{-mod}(\text{Set}[\mathbb{T}])$ and the unique (up to isomorphism) extension E of M along \mathbb{E} . By Lemma 2.3.4, $M + E$ is a universal model of $\mathbb{T} + \mathbb{E}$. Now, for every relation symbol $R \subseteq \vec{A}$ introduced in \mathbb{E} , $\llbracket R \rrbracket_{M+E}$ is a subobject of $\llbracket A_1 \rrbracket_M \times \cdots \times \llbracket A_n \rrbracket_M$ and since M is universal, [4, Theorem 6.1.3] tells us that there is a formula ϕ_R of \mathbb{T} with $\llbracket \phi_R \rrbracket_M = \llbracket R \rrbracket_{M+E}$. Similarly, for every new function symbol f we find a provably functional formula ψ_f of \mathbb{T} such that $\llbracket \psi_f \rrbracket_M$ is the graph of $\llbracket f \rrbracket_{M+E}$. That is, the axioms for defining R by ϕ_R and f by ψ_f are fulfilled for $M + E$ and therefore provable in $\mathbb{T} + \mathbb{E}$. Thus, we have $\mathbb{E} = \mathbb{E}_1 + \mathbb{E}_2$ where \mathbb{E}_1 is an extension by definitions and \mathbb{E}_2 is a quotient. But since both \mathbb{E}_1 and $\mathbb{E}_1 + \mathbb{E}_2$ are equivalence extensions, \mathbb{E}_2 must be one too, and we have (up to syntactic equivalence) $\mathbb{E}_2 = \emptyset$ and $\mathbb{E} = \mathbb{E}_1$. \square

The following lemma is about “reverting” an extension \mathbb{E} of \mathbb{T} by applying another extension \mathbb{E}' , subject to extensibility of the universal \mathbb{T} -model along \mathbb{E} .

Lemma 2.3.8. *Let $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ be a universal model and E an extension of M along some theory extension \mathbb{E} . Then there is a localic extension \mathbb{E}' of $\mathbb{T} + \mathbb{E}$ and a model extension E' of $M + E$ along \mathbb{E}' such that $(M + E + E') \in (\mathbb{T} + \mathbb{E} + \mathbb{E}')\text{-mod}(\mathcal{E})$ is again universal. (In particular, $\mathbb{E} + \mathbb{E}'$ is an equivalence extension.) If \mathbb{E} is localic, \mathbb{E}' can be chosen as a quotient; if \mathbb{E} is a quotient, $\mathbb{E}' = \emptyset$ fits the bill.*

$$\begin{array}{c} \mathbb{T} + \mathbb{E} + \mathbb{E}' \\ \left. \begin{array}{c} | \\ \mathbb{T} + \mathbb{E} \\ | \\ \mathbb{T} \end{array} \right\} \sim \end{array}$$

Proof. Fix a classifying topos $\mathcal{E}_{\mathbb{T}+\mathbb{E}}$ for $\mathbb{T}+\mathbb{E}$. We have the forgetful geometric morphism $u : \mathcal{E}_{\mathbb{T}+\mathbb{E}} \rightarrow \mathcal{E}$, and $M + E \in (\mathbb{T} + \mathbb{E})\text{-mod}(\mathcal{E})$ corresponds to a section (up to isomorphism) $s : \mathcal{E} \rightarrow \mathcal{E}_{\mathbb{T}+\mathbb{E}}$ of u . As such, s is localic (since $u \circ s$ is). By Theorem 2.2.2, it is therefore presented by some localic extension \mathbb{E}' of $\mathbb{T} + \mathbb{E}$, that is, we have a universal model of $\mathbb{T} + \mathbb{E} + \mathbb{E}'$ in \mathcal{E} extending $M + E$, as required.

The two special cases follow since a section of a localic geometric morphism is an embedding and a section of an embedding is an equivalence, but we can also show them more directly. If $\mathbb{E} = \mathbb{Q}$ is a quotient, then the existence of E just means that the new axioms are fulfilled in M . But this means that they were already provable in \mathbb{T} and $M = M + E$ is indeed a universal model of $\mathbb{T} + \mathbb{Q}$. For \mathbb{E} localic, we have to deal with the new relation symbols R and function symbols f . From [4, Theorem 6.1.3] we know that $\llbracket R(\vec{x}) \rrbracket_{M+E} = \llbracket \phi_R(\vec{x}) \rrbracket_M$ and $\llbracket f(\vec{x}) = y \rrbracket_{M+E} = \llbracket \phi_f(\vec{x}, y) \rrbracket_M$ for some formulas ϕ_R, ϕ_f , where ϕ_f is provably functional. So taking for \mathbb{E}' the axioms $R(\vec{x}) \dashv\vdash_{\vec{x}} \phi_R(\vec{x})$ and $f(\vec{x}) = y \dashv\vdash_{\vec{x}, y} \phi_f(\vec{x}, y)$, we obtain as $\mathbb{E} + \mathbb{E}'$ an extension by definitions. \square

As a corollary, we find that an equivalence between two theories, and more generally an equivalence between two extensions of some base theory (meaning that they present the same geometric morphism), can always be captured in a syntactic way.

Corollary 2.3.9. *Let $\mathbb{E}_1, \mathbb{E}_2$ be two equivalent extensions of a theory \mathbb{T} , that is, there exists a model $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ with extensions to universal models $M + E_1$ and $M + E_2$ of $\mathbb{T} + \mathbb{E}_1$ respectively $\mathbb{T} + \mathbb{E}_2$ in the same topos \mathcal{E} . Then there is a localic extension $\mathbb{E}_{1,2}$ of $\mathbb{T} + \mathbb{E}_1 + \mathbb{E}_2$ such that both $\mathbb{E}_2 + \mathbb{E}_{1,2}$ and $\mathbb{E}_1 + \mathbb{E}_{1,2}$ are equivalence extensions (of $\mathbb{T} + \mathbb{E}_1$ respectively $\mathbb{T} + \mathbb{E}_2$)*

$$\begin{array}{ccc}
 & \mathbb{T} + \mathbb{E}_1 + \mathbb{E}_2 + \mathbb{E}_{1,2} & \\
 \sim & \swarrow & \searrow \sim \\
 \mathbb{T} + \mathbb{E}_1 & & \mathbb{T} + \mathbb{E}_2 \\
 & \searrow & \swarrow \\
 & \mathbb{T} &
 \end{array}$$

and there is a model extension $E_{1,2}$ along $\mathbb{E}_{1,2}$ such that $M_0 + E_1 + E_2 + E_{1,2}$ is universal. If \mathbb{E}_1 or \mathbb{E}_2 is localic, then $\mathbb{E}_{1,2} = \mathbb{Q}_{1,2}$ can be chosen as a quotient extension.

Proof. Regard \mathbb{E}_2 as an extension of $\mathbb{T} + \mathbb{E}_1$ for the moment, then we have an extension E_2 of the universal model $M + E_1$, so by Lemma 2.3.8 there exists a localic extension $\mathbb{E}_{1,2}$ of $\mathbb{T} + \mathbb{E}_1 + \mathbb{E}_2$ together with a model extension $E_{1,2}$ of $M + E_1 + E_2$ such that $M + E_1 + E_2 + E_{1,2}$ is universal. Then, by Lemma 2.3.4, both $\mathbb{E}_2 + \mathbb{E}_{1,2}$ and $\mathbb{E}_1 + \mathbb{E}_{1,2}$ are equivalence extensions, as required. If \mathbb{E}_1 is localic, we get a quotient extension $\mathbb{E}_{1,2} = \mathbb{Q}_{1,2}$ from Lemma 2.3.8, and if \mathbb{E}_2 is localic, we swap the two. \square

A completely different proof of Corollary 2.3.9 in the absolute case $\mathbb{T}_0 = \emptyset$ can be found in [8, Theorem 5.1].

Definition 2.3.10. We call an extension $\mathbb{E}_{1,2}$ as in Corollary 2.3.9 a *diagonal extension* for \mathbb{E}_1 and \mathbb{E}_2 over \mathbb{T} , because it presents the diagonal geometric morphism of the pullback topos

$$\text{Set}[\mathbb{T} + \mathbb{E}_1 + \mathbb{E}_2] = \text{Set}[\mathbb{T} + \mathbb{E}_1] \times_{\text{Set}[\mathbb{T}]} \text{Set}[\mathbb{T} + \mathbb{E}_2] = \mathcal{E} \times_{\text{Set}[\mathbb{T}]} \mathcal{E}.$$

Diagonal extensions are not unique. For example, if $\mathbb{T}_1 = \langle p_1 \rangle + \langle p_2 \rangle$ and $\mathbb{T}_2 = \langle p_3 \rangle + \langle p_4 \rangle$ both consist of two proposition symbols, there are clearly two different diagonal quotient extensions for \mathbb{T}_1 and \mathbb{T}_2 over \emptyset . Corollary 2.3.9 produces a diagonal extension in accordance with the two universal models $M_0 + E_1$ and $M_0 + E_2$ living in the same topos.

Another easy consequence of Lemma 2.3.8 is that any object of the classifying topos of a theory can be introduced into the theory as a new sort by means of an equivalence extension.

Corollary 2.3.11. *Let $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ be a universal model and let $X \in \mathcal{E}$ be any object. Then there is an equivalence extension \mathbb{E} of \mathbb{T} containing exactly one new sort A , such that the unique (up to unique isomorphism) model extension E of M along \mathbb{E} interprets A as X .*

Proof. Apply Lemma 2.3.8 to the extension \mathbb{E}_1 adding nothing but the sort A (and the model extension E_1 given by $\llbracket A \rrbracket_{E_1} := X$) to obtain a localic extension \mathbb{E}_2 , and set $\mathbb{E} := \mathbb{E}_1 + \mathbb{E}_2$. \square

2.4 Some examples of equivalence extensions

Giving examples of localic equivalence extensions does not seem necessary after Proposition 2.3.7, but here are some non-localic equivalence extensions that will be of use later.

Example 2.4.1. Let A be a sort of a geometric theory \mathbb{T} and let $\phi(x)$ be a geometric formula in the context $x : A$. Then there is an equivalence extension of \mathbb{T} consisting of a sort S_ϕ , a function symbol $\iota : S_\phi \rightarrow A$ and the axioms

$$\iota(x) = \iota(y) \vdash_{x,y:S_\phi} x = y \quad \text{and} \quad \exists x : S_\phi. \iota(x) = y \dashv\vdash_{y:A} \phi(y),$$

the first of which forces the interpretation of ι in a model to be a monomorphism, while the second ensures that $\llbracket S_\phi \rrbracket \hookrightarrow \llbracket A \rrbracket$ is the same subobject as $\llbracket \phi \rrbracket \hookrightarrow \llbracket A \rrbracket$.

Example 2.4.2. Let A be a sort of a geometric theory \mathbb{T} and let $x \sim x'$ be a geometric formula in the context $x, x' : A$ such that the usual axioms of an equivalence relation (which are Horn sequents) are provable for \sim . Then there is an equivalence extension of \mathbb{T} consisting of a sort A/\sim , a function symbol $\pi : A \rightarrow A/\sim$ and the axioms

$$\top \vdash_{y:A/\sim} \exists x : A. \pi(x) = y \quad \text{and} \quad \pi(x) = \pi(x') \dashv\vdash_{x,x':A} x \sim x'.$$

These force the interpretation $\llbracket A \rrbracket \rightarrow \llbracket A/\sim \rrbracket$ of π to be an epimorphism with kernel pair $\llbracket \sim \rrbracket \rightrightarrows \llbracket A \rrbracket$.

Remark 2.4.3. An intermediate notion between syntactic equivalence and Morita equivalence is the notion of (geometric) *bi-interpretability*, see [4, Definition 2.1.13]. Two geometric theories are bi-interpretable if and only if their syntactic sites are equivalent categories. So while this notion is still syntactic in nature, it does not assume any previously given relation between the signatures of the two theories. For example, a sort A of the first theory, which is represented in the syntactic site by the object $\{x : A. \top\}$, can correspond to any formula in context $\{\vec{y} : \vec{B}. \phi(\vec{y})\}$ of the second theory.

The equivalence extension in Example 2.4.1 induces a bi-interpretation, interpreting the new sort S_ϕ by the formula in context $\{x : A. \phi(x)\}$ of \mathbb{T} . But already Example 2.4.2 shows that bi-interpretability is a stronger condition than Morita equivalence, since the formula in context $\{x : A/\sim. \top\}$ can not be expressed as any formula in context of \mathbb{T} .

Example 2.4.4. Given a set A , define a theory \underline{A} (named after its only sort) consisting of a sort \underline{A} , constant symbols $c_a : \underline{A}$ for every element $a \in A$ and the axioms

$$c_a = c_{a'} \vdash_{\square} \perp \quad \text{for } a \neq a' \in A, \quad \top \vdash_{x:\underline{A}} \bigvee_{a \in A} (x = c_a).$$

One can check that the unique model (up to unique isomorphism) in any Grothendieck topos is the constant sheaf associated to the set A , which is also denoted \underline{A} . The theory \underline{A} is therefore Morita-equivalent to the empty theory, and adding \underline{A} to any given theory is an equivalence extension. In other words, we can always *import* a set A into our theory without changing it up to Morita-equivalence.

If we have a function like $f : A \rightarrow B$ or a relation like $R \subseteq A$, we can import it together with the respective sets. Namely, after adding a function symbol $\underline{f} : \underline{A} \rightarrow \underline{B}$ or a relation symbol $\underline{R} \subseteq \underline{A}$, the axioms

$$\top \vdash_{\square} \underline{f}(c_a) = c_{f(a)} \quad \text{for } a \in A$$

respectively

$$\top \vdash_{\square} \underline{R}(c_a) \quad \text{for } a \in R, \quad \underline{R}(c_a) \vdash_{\square} \perp \quad \text{for } a \in A \setminus R$$

produces an extension by definitions in presence of the axioms of \underline{A} .

Another perspective on this is that if we have any model of a geometric theory \mathbb{T} in Set , then by Lemma 2.3.8 there is a localic extension to a universal model in Set , which then yields, by pulling back along the unique geometric morphism to Set , the unique model of the extended theory in any topos.

3 Gluing classifying toposes

3.1 Introduction

In this section we explicitly construct a geometric theory classified by a given topos from a cover by open subtoposes with known classified theories.

We already saw in Section 2 that taking the product of two toposes, regarded as generalized spaces, corresponds to the simplest possible operation involving two unrelated theories, namely forming the sum $\mathbb{T}_1 + \mathbb{T}_2$. The present topic is a generalization of the dual question what the coproduct of two toposes classifies. This is much less obvious, since a geometric morphism $\mathcal{E} \rightarrow \text{Set}[\mathbb{T}_1] \amalg \text{Set}[\mathbb{T}_2]$ will neither define a \mathbb{T}_1 -model nor a \mathbb{T}_2 -model in \mathcal{E} . Instead, it first of all defines a decomposition of \mathcal{E} into two clopen subtoposes, and then a \mathbb{T}_1 -model in one and a \mathbb{T}_2 -model in the other of these subtoposes.

Another good example is the big Zariski topos of the projective line \mathbb{P}_K^1 over a ring K . The big Zariski topos X_{Zar} of an affine scheme $X = \text{Spec } R$ classifies the geometric theory of local R -algebras. Now, \mathbb{P}_K^1 is not affine, but it can be covered by two copies of the affine line, $\text{Spec } K[X]$ and $\text{Spec } K[Y]$, such that the intersection is $\text{Spec } K[X, Y]/(XY - 1)$. We will see that this induces an open cover of the big Zariski topos $(\mathbb{P}_K^1)_{\text{Zar}}$ by open subtoposes $(\text{Spec } K[X])_{\text{Zar}}$ and $(\text{Spec } K[Y])_{\text{Zar}}$, which both classify local K -algebras with one distinguished element. This suggests that $(\mathbb{P}_K^1)_{\text{Zar}}$ classifies local K -algebras which are equipped with an element X or with an element Y , where the two possibilities are not mutually exclusive but rather, their intersection is described by the condition $XY = 1$. A formulation of this idea as a geometric theory, which indeed presents $(\mathbb{P}^1)_{\text{Zar}}$, is given in Proposition 3.7.5.

3.2 Conditional extensions

If a topos \mathcal{E} has an open subtopos $\mathcal{E}_1 \subseteq \mathcal{E}$ which classifies some theory \mathbb{T}_1 , then a geometric morphism $f : \tilde{\mathcal{E}} \rightarrow \mathcal{E}$ does not give us a model of \mathbb{T}_1 in $\tilde{\mathcal{E}}$, so to find a classified theory for \mathcal{E} , we should not look among extensions of \mathbb{T}_1 . However, f does give us a model of \mathbb{T}_1 in some open subtopos of $\tilde{\mathcal{E}}$, namely in the preimage of \mathcal{E}_1 . We now show how to capture syntactically the requirement of “a model in some open subtopos”.

To avoid technical complications, we exclude function symbols from our discussion. Recall that a function symbol $f : \vec{A} \rightarrow B$ can be considered an abbreviation for a relation symbol $R \subseteq \vec{A} \times B$ together with axioms ensuring that R is provably functional, as long as we don't mind replacing axioms with nested function applications like $g(f(x)) = z$ by versions with auxiliary variables like $\exists y. (f(x) = y) \wedge (g(y) = z)$. That is, function symbols can be considered “syntactic sugar”.

Definition 3.2.1. Let \mathbb{T} be a geometric theory, ϕ a closed formula of \mathbb{T} and \mathbb{E} an extension (without function symbols) of $\mathbb{T} + \phi$. We define the *conditional extension* \mathbb{E}/ϕ of \mathbb{T} to consist of the following.

- For every sort $S \in \mathbb{E}$ -Sort of \mathbb{E} , a sort S and the axiom $\top \vdash_{x:S} \phi$.

- For every relation symbol $R \subseteq S_1 \times \cdots \times S_n$ of \mathbb{E} , a relation symbol $R \subseteq S_1 \times \cdots \times S_n$ and the axiom $R(\vec{x}) \vdash_{\vec{x}:\vec{S}} \phi$.
- For every axiom $\psi_1 \vdash_{\Gamma} \psi_2$ of \mathbb{E} , the axiom $\psi_1 \wedge \phi \vdash_{\Gamma} \psi_2$.

If the theory $\mathbb{T} + \phi + \mathbb{E}$ asks us to specify a model of \mathbb{T} that satisfies ϕ and furthermore a model extension along \mathbb{E} , then the theory $\mathbb{T} + \mathbb{E}/\phi$ instead asks us to specify a model of \mathbb{T} and a model extension along \mathbb{E} *in case* our \mathbb{T} -model happens to satisfy ϕ — or, more geometrically speaking, *wherever* it satisfies ϕ .

Lemma 3.2.2. *The assignment $\mathbb{E} \mapsto \mathbb{E}/\phi$ is well-defined with respect to syntactic equivalence. Furthermore, we have the following syntactic equivalences.*

$$(i) \quad \mathbb{T} + \mathbb{E}/\phi + \phi = \mathbb{T} + \phi + \mathbb{E}$$

$$(ii) \quad \mathbb{T} + \mathbb{E}_1/\phi + \mathbb{E}_2/\phi = \mathbb{T} + (\mathbb{E}_1 + \mathbb{E}_2)/\phi$$

Proof. To show that the construction is well-defined, let $\psi \vdash_{\Gamma} \chi$ be a geometric sequent which is provable in $\mathbb{T} + \phi + \mathbb{E}$. If it was added as an axiom to \mathbb{E} , the axiom $\psi \wedge \phi \vdash_{\Gamma} \chi$ would be added to \mathbb{E}/ϕ , so we must show that this sequent is already provable in $\mathbb{T} + \mathbb{E}/\phi$. This is equivalent to showing that $\psi \vdash_{\Gamma} \chi$ is provable in $\mathbb{T} + \mathbb{E}/\phi + \phi$, so we are done if (i) is true. For (i), we observe that the additional axioms of \mathbb{E}/ϕ are indeed trivial and $\psi_1 \wedge \phi \vdash_{\Gamma} \psi_2$ is indeed equivalent to $\psi_1 \vdash_{\Gamma} \psi_2$ if our theory already contains ϕ as an axiom. For (ii), Definition 3.2.1 produces the exact same axiomatization for both sides. \square

Proposition 3.2.3. *Let \mathbb{E} be an extension of $\mathbb{T} + \phi$, $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ and $\mathcal{F} := \mathcal{E}_{o(\llbracket \phi \rrbracket_M)}$. Then there is an equivalence of categories*

$$\mathbb{E}\text{-mod}(\mathcal{F}, M|_{\mathcal{F}}) \simeq (\mathbb{E}/\phi)\text{-mod}(\mathcal{E}, M).$$

Proof. The open inclusion geometric morphism $i : \mathcal{F} \rightarrow \mathcal{E}$ admits a further (full and faithful) left adjoint $i_! : \mathcal{F} \rightarrow \mathcal{E}$ with essential image \mathcal{E}/U , where $U := \llbracket \phi \rrbracket_M$. The composition $i_! \circ i^*$ is $X \mapsto X \times U$.

$$\mathcal{E}/U \simeq \mathcal{F} \begin{array}{c} \xrightarrow{i_!} \\ \perp \\ \xleftarrow{i^*} \end{array} \mathcal{E}$$

For a sort S of \mathbb{E} , the axiom $\top \vdash_{x:S} \phi$ of \mathbb{E}/ϕ means precisely that for every $E \in (\mathbb{E}/\phi)\text{-mod}(M)$, the object $\llbracket S \rrbracket_{M+E}$ must lie in the full subcategory \mathcal{E}/U , so we have established the equivalence for the case that \mathbb{E} contains only sorts. The quasi-inverse is of course the restriction functor $i^* : \mathcal{E} \rightarrow \mathcal{F}$, applied to the interpretation of every sort. For a relation symbol $R \subseteq \vec{S}$ of \mathbb{E} (where the list \vec{S} may contain sorts from both \mathbb{T} and \mathbb{E}), the axiom $R(\vec{x}) \vdash_{\vec{x}:\vec{S}} \phi$ similarly means $\llbracket R \rrbracket_{M+E} \subseteq \llbracket \vec{S} \rrbracket_{M+E} \times U$, so that $\llbracket R \rrbracket_{M+E}$ corresponds to a subobject of $i^*(\llbracket \vec{S} \rrbracket_{M+E}) = \llbracket \vec{S} \rrbracket_{M|_{\mathcal{F}+E}|_{\mathcal{F}}}$. Finally, the modified axiom $\psi_1 \wedge \phi \vdash_{\Gamma} \psi_2$ (or equivalently $\psi_1 \wedge \phi \vdash_{\Gamma} \psi_2 \wedge \phi$) is satisfied by $M + E$ if and only if $\llbracket \psi_1 \rrbracket \times U \subseteq \llbracket \psi_2 \rrbracket \times U$, that is, if and only if $\psi_1 \vdash_{\Gamma} \psi_2$ is satisfied by $M|_{\mathcal{F} + E}|_{\mathcal{F}}$. \square

For a model $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ and an extension $E \in \mathbb{E}\text{-mod}(M|_{\mathcal{E}_o(\mathbb{1}_M)})$, the corresponding extension of M along \mathbb{E}/ϕ will be denoted E/ϕ . That means that we have

$$(M + E/\phi)|_{\mathcal{E}_o(\mathbb{1}_M)} \cong M|_{\mathcal{E}_o(\mathbb{1}_M)} + E$$

as models of $\mathbb{T} + \mathbb{E}/\phi + \phi = \mathbb{T} + \phi + \mathbb{E}$.

Lemma 3.2.4. *An extension \mathbb{E} of $\mathbb{T} + \phi$ is an equivalence extension if and only if \mathbb{E}/ϕ is an equivalence extension of \mathbb{T} .*

Proof. If \mathbb{E}/ϕ is an equivalence extension of \mathbb{T} , then it is also an equivalence extension of $\mathbb{T} + \phi$. But by Lemma 3.2.2 (i), \mathbb{E}/ϕ is syntactically equivalent to \mathbb{E} as an extension of $\mathbb{T} + \phi$.

Conversely, if \mathbb{E} is an equivalence extension of $\mathbb{T} + \phi$, then by Lemma 2.3.2, and Proposition 3.2.3 every \mathbb{T} -model admits a unique (up to isomorphism) model extension along \mathbb{E}/ϕ , so we are done by the other direction of Lemma 2.3.2. \square

3.3 Systems of theory extensions

Given two equivalence extensions $\mathbb{E}_1, \mathbb{E}_2$ of a theory \mathbb{T} , their sum $\mathbb{E}_1 + \mathbb{E}_2$ is again an equivalence extension of \mathbb{T} . The same is true if \mathbb{E}_2 is not an extension of \mathbb{T} but of $\mathbb{T} + \mathbb{E}_1$, that is, if \mathbb{E}_2 depends on \mathbb{E}_1 . To formulate such statements in greater generality, we first need to clarify in which ways an extension can be built up from smaller extensions, possibly with dependencies among them.

Definition 3.3.1. Let \mathbb{T} be a geometric theory. A *system of extensions* over \mathbb{T} is a family $(\mathbb{E}_i)_{i \in I}$, indexed by some partially ordered set I , where each \mathbb{E}_i consists of a set of sorts, sets of relation and function symbols — whose signatures may contain sorts from \mathbb{T} and all \mathbb{E}_j with $j \leq i$, treated disjointly — and a set of (geometric) axioms — which may again use all sorts and symbols from \mathbb{T} and \mathbb{E}_j with $j \leq i$, treated disjointly. Taking the disjoint union of the sorts, symbols and axioms of all \mathbb{E}_i thus yields an extension of \mathbb{T} , which we denote $\sum_{i \in I} \mathbb{E}_i$.

For a subset $J \subseteq I$ of the partially ordered set I , we use the notation

$$J \downarrow := \{i \in I \mid \exists j \in J : i \leq j\} \quad \text{and} \quad J \downarrow \downarrow := J \downarrow \setminus J.$$

Given an initial segment $J \subseteq I$ (that is, $J \downarrow = J$), we can of course restrict a system $(\mathbb{E}_i)_{i \in I}$ to J , obtaining a system $(\mathbb{E}_i)_{i \in J}$ over the same base theory \mathbb{T} . In particular, we see that for every $i \in I$, \mathbb{E}_i is an extension of $\mathbb{T} + \sum_{j < i} \mathbb{E}_j$. More generally, the index set I can be restricted to any *inward closed* subset $J \subseteq I$ (if $i \leq j \leq k$ and $i, k \in J$ then $j \in J$), resulting in a system $(\mathbb{E}_i)_{i \in J}$ over $\mathbb{T} + \sum_{i \in J \downarrow \downarrow} \mathbb{E}_i$ instead of \mathbb{T} .

We can also push a system $(\mathbb{E}_i)_{i \in I}$ over \mathbb{T} forward along an order-preserving map $f : I \rightarrow J$ to another partially ordered set J by setting $\tilde{\mathbb{E}}_j := \sum_{i \in f^{-1}(j)} \mathbb{E}_i$. Note that $f^{-1}(j)$ is inward closed and $\tilde{\mathbb{E}}_j$ is an extension of $\mathbb{T} + \sum_{i \in f^{-1}(j) \downarrow \downarrow} \mathbb{E}_i$ and therefore also of $\mathbb{T} + \sum_{j' < j} \tilde{\mathbb{E}}_{j'}$.

Definition 3.3.2. Let $(\mathbb{E}_i)_{i \in I}$ be a system of extensions over a theory \mathbb{T} . A *system of model extensions* $(E_i)_{i \in I}$ for $(\mathbb{E}_i)_{i \in I}$ in a Grothendieck topos \mathcal{E} over some $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ is just a model extension $\sum_{i \in I} E_i$ of M along $\sum_{i \in I} \mathbb{E}_i$, regarded as a family of model extensions E_i of $M + \sum_{j < i} E_j$ along \mathbb{E}_i .

For a family of subtoposes $(\mathcal{E}_i)_{i \in I}$ of \mathcal{E} with $\mathcal{E}_i \subseteq \mathcal{E}_j$ for $j < i$, a system of model extensions *in the* \mathcal{E}_i is similarly a family $(E_i)_{i \in I}$, where each E_i is an extension of $M|_{\mathcal{E}_i} + \sum_{j < i} E_j|_{\mathcal{E}_i}$ along \mathbb{E}_i . If $M|_{\mathcal{E}_i} + \sum_{j < i} E_j|_{\mathcal{E}_i}$ is a universal model of $\mathbb{T} + \sum_{j < i} \mathbb{E}_j$ for every $i \in I$, then we call $(\mathbb{E}_i, E_i)_{i \in I}$ a *system of presentations* of the \mathcal{E}_i over $(\mathbb{T}$ and) M .

The following lemma is our general formulation of how equivalence extensions can be built up from smaller equivalence extensions. We will also use it (in Corollary 3.3.5 below) to clarify the role of systems of presentations which all present the same topos.

Lemma 3.3.3. *Let $(\mathbb{E}_i)_{i \in I}$ be a system over \mathbb{T} where the partial order I is well-founded. Then the assertions*

(i) *For every $i \in I$, \mathbb{E}_i is an equivalence extension of $\mathbb{T} + \sum_{j < i} \mathbb{E}_j$.*

(ii) *For every $i \in I$, $\sum_{j \leq i} \mathbb{E}_j$ is an equivalence extension of \mathbb{T} .*

are equivalent and they imply:

(iii) *$\sum_{i \in I} \mathbb{E}_i$ is an equivalence extension of \mathbb{T} .*

Proof. (i) \Rightarrow (iii). We show this implication first, because it will be used for the others. To see that the functor

$$U_{\sum_{i \in I} \mathbb{E}_i} : (\mathbb{T} + \sum_{i \in I} \mathbb{E}_i)\text{-mod}(\mathcal{E}) \rightarrow \mathbb{T}\text{-mod}(\mathcal{E})$$

is an equivalence, start by taking two models M, M' of $\mathbb{T} + \sum_{i \in I} \mathbb{E}_i$ in \mathcal{E} and homomorphisms $f, g : M \rightarrow M'$ with $U_{\sum_{i \in I} \mathbb{E}_i}(f) = U_{\sum_{i \in I} \mathbb{E}_i}(g)$. Then we can show $f = g$ by well-founded induction over $i \in I$ using the faithfulness of each

$$U_{\mathbb{E}_i} : (\mathbb{T} + \sum_{j < i} \mathbb{E}_j)\text{-mod}(\mathcal{E}) \rightarrow (\mathbb{T} + \sum_{j < i} \mathbb{E}_j)\text{-mod}(\mathcal{E}).$$

Given instead only $f_0 : U_{\sum_{i \in I} \mathbb{E}_i}(M) \rightarrow U_{\sum_{i \in I} \mathbb{E}_i}(M')$, we use the fullness of the $U_{\mathbb{E}_i}$ to construct $f : M \rightarrow M'$ with $U_{\sum_{i \in I} \mathbb{E}_i}(f) = f_0$ by well-founded recursion. Finally, let $M_0 \in \mathbb{T}\text{-mod}(\mathcal{E})$ and notice that the $U_{\mathbb{E}_i}$ are not only essentially surjective but strictly surjective on objects (since they are isofibrations). This means we can again use well-founded recursion to construct a model extension $\sum_{i \in I} E_i$ of M_0 and therefore a (strict) preimage $M_0 + \sum_{i \in I} E_i$ of M_0 under $U_{\sum_{i \in I} \mathbb{E}_i}$.

(i) \Rightarrow (ii). Fix $i \in I$. Then this follows immediately by applying (i) \Rightarrow (iii) to the restricted system $(\mathbb{E}_j)_{j < i}$.

(ii) \Rightarrow (i). By well-founded induction, let $i \in I$ with \mathbb{E}_j an equivalence extension for all $j < i$. Then by (i) \Rightarrow (iii) applied to $(\mathbb{E}_j)_{j < i}$, $\sum_{j < i} \mathbb{E}_j$ is an equivalence extension. But since $\sum_{j \leq i} \mathbb{E}_j = \sum_{j < i} \mathbb{E}_j + \mathbb{E}_i$ is an equivalence extension by assumption, \mathbb{E}_i must be one too. \square

Remark 3.3.4. The assumption that I is well-founded is necessary in Lemma 3.3.3. Indeed, we can take \mathbb{T} to be the empty theory and consider the system of theory extensions $(\mathbb{E}_n)_{n \in \mathbb{Z}}$ where every \mathbb{E}_n is the quotient extension consisting of the contradictory axiom $\top \vdash_{\square} \perp$. Then none of the $\mathbb{T} + \sum_{m \leq n} \mathbb{E}_m$ is equivalent to \mathbb{T} , and neither is $\mathbb{T} + \sum_{n \in \mathbb{Z}} \mathbb{E}_n$, but every \mathbb{E}_n is an equivalence extension of $\mathbb{T} + \sum_{m < n} \mathbb{E}_m$. If we insist on treating syntactically equivalent theories as equal, we can even say that every \mathbb{E}_n is the *empty* extension of its respective base theory. In this sense, a system of theory extensions $(\mathbb{E}_i)_{i \in I}$ is not fully determined by the individual extensions \mathbb{E}_i , for I not well-founded. But we would rather argue that the notion of knowing an extension without knowing its base theory is ill-defined. (In the language of type theory, the type of extensions is inherently a dependent type.)

Corollary 3.3.5. *Let $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ be universal. Then a system of extensions $(\mathbb{E}_i, E_i)_{i \in I}$ over (\mathbb{T}, M) in \mathcal{E} with I well-founded is a system of presentations of \mathcal{E} if and only if all \mathbb{E}_i are equivalence extensions. And in this case, $M + \sum_{i \in I} E_i \in (\mathbb{T} + \sum_{i \in I} \mathbb{E}_i)\text{-mod}(\mathcal{E})$ is also universal.*

Proof. By Lemma 2.3.4, the universality of the models $M + \sum_{j \leq i} E_j$ for all $i \in I$ is just condition (ii) of Lemma 3.3.3 and the universality of $M + \sum_{i \in I} E_i$ is condition (iii) of Lemma 3.3.3. \square

3.4 Systems of conditional extensions

We now briefly describe how to apply the construction of conditional extensions to systems of theory extensions. Given a system $(\mathbb{E}_i)_{i \in I}$ over \mathbb{T} and a family of closed geometric formulas ϕ_i of \mathbb{T} , we can of course blindly apply the rules of Definition 3.2.1 to every \mathbb{E}_i , and we do get a new system of extensions

$$(\mathbb{E}_i/\phi_i)_{i \in I}$$

over \mathbb{T} , simply because the signature of \mathbb{E}_i/ϕ_i is the same as that of \mathbb{E}_i . But then we can not say that the extension \mathbb{E}_i/ϕ_i of $\mathbb{T} + \sum_{j < i} \mathbb{E}_j/\phi_j$ is an honest conditional extension, since there is no sensible way to regard \mathbb{E}_i as an extension of $\mathbb{T} + \sum_{j < i} \mathbb{E}_j/\phi_j + \phi_i$. For example, fixing some \mathbb{T} , \mathbb{E}_1 and ϕ_1 and using $\phi_2 = \top$, the operation of mapping an extension \mathbb{E}_2 of $\mathbb{T} + \phi_1 + \mathbb{E}_1$ to $\mathbb{T} + \mathbb{E}_1/\phi_1 + \mathbb{E}_2$ is not well-defined with respect to syntactic equivalence.

To remedy this, we impose the condition that

$$\phi_i \vdash_{\square} \phi_j \quad \text{for } j \leq i$$

is provable in \mathbb{T} . Then we can regard each \mathbb{E}_i as an extension of

$$\mathbb{T} + \sum_{j < i} \mathbb{E}_j/\phi_j + \phi_i = \mathbb{T} + \sum_{j < i} \mathbb{E}_j + \phi_i,$$

and obtain the system $(\mathbb{E}_i/\phi_i)_{i \in I}$ of conditional extensions.

In Definition 3.2.1, we treated \mathbb{E} as an extension of $\mathbb{T} + \phi$, not of \mathbb{T} , which is justified in view of Lemma 3.2.4. This refinement is dropped here in order to be able to use our definition of a system of theory extensions without modification. In our application (see Theorem 3.5.4 below), the extensions \mathbb{E}_i are such that ϕ_i is provable from $\mathbb{T} + \sum_{j \leq i} \mathbb{E}_j$.

We also want to be able to construct model extensions, to be denoted as

$$M + \sum_{i \in I} E_i / \phi_i,$$

for systems of conditional extensions, as we did in Proposition 3.2.3 for a single conditional extension. Let $(E_i)_{i \in I}$ be a family of model extensions for $(\mathbb{E}_i)_{i \in I}$, over some $M \in \mathbb{T}\text{-mod}(\mathcal{E})$, in the subtoposes

$$\mathcal{E}_i := \mathcal{E}_{o(\llbracket \phi_i \rrbracket_M)}.$$

This makes sense because our requirement on the ϕ_i ensures $\mathcal{E}_i \subseteq \mathcal{E}_j$ for $j \leq i$. Let $\iota_i : \mathcal{E}_i \rightarrow \mathcal{E}$ be the open embeddings, and also $\iota_i^j : \mathcal{E}_i \rightarrow \mathcal{E}_j$ for $j \leq i$.

$$\begin{array}{ccc} \mathcal{E}_j & \xrightarrow{\iota_j} & \mathcal{E} \\ \uparrow \iota_i^j & \nearrow \iota_i & \\ \mathcal{E}_i & & \end{array}$$

Then, like in Proposition 3.2.3, applying $(\iota_i)_!$ to the interpretations of the sorts of \mathbb{E}_i in E_i produces objects in $\mathcal{E}/\llbracket \phi_i \rrbracket_M$, as required for a model extension of M along $\sum_{i \in I} \mathbb{E}_i / \phi_i$. The signatures of the relation symbols of \mathbb{E}_i can contain sorts S from \mathbb{E}_i , sorts S' from various \mathbb{E}_j with $j \leq i$ and sorts S'' from \mathbb{T} , so the interpretation of such a symbol is a subobject of an object like

$$\llbracket S \rrbracket \times (\iota_i^j)^* \llbracket S' \rrbracket \times (\iota_i)^* \llbracket S'' \rrbracket \in \mathcal{E}_i.$$

Applying $(\iota_i)_!$ yields precisely a subobject of $(\iota_i)_! \llbracket S \rrbracket \times (\iota_j)_! \llbracket S' \rrbracket \times \llbracket S'' \rrbracket \times \llbracket \phi_i \rrbracket$, since

$$(\iota_i)_! (\iota_i^j)^* \llbracket S' \rrbracket = (\iota_i)_! (\iota_i)^* (\iota_j)_! \llbracket S' \rrbracket = (\iota_j)_! \llbracket S' \rrbracket \times \llbracket \phi_i \rrbracket.$$

Also, the modified axioms of \mathbb{E}_i in \mathbb{E}_i / ϕ_i hold for the structure in \mathcal{E} defined this way because they can be tested after pulling back to \mathcal{E}_i , as before. In summary, we have constructed a model extension $\sum_{i \in I} E_i / \phi_i$ of M (in the topos \mathcal{E}), such that for every i , the extension E_i / ϕ_i is obtained from the model extension E_i (in the topos \mathcal{E}_i) as in Proposition 3.2.3, in accordance with our previous use of the notation E / ϕ .

3.5 Syntactic presentation of a glued topos

We need a lemma about testing the property of a geometric morphism to be an equivalence on an open cover of the target topos. Or rather, we need its reformulation in terms of models of theories (Corollary 3.5.3 below), saying that the property of a model to be universal can be tested on an “open cover of the theory”.

Lemma 3.5.1. *Let $f : \mathcal{E} \rightarrow \mathcal{F}$ be a geometric morphism and let a cover $\mathcal{F} = \bigcup_{i \in I} \mathcal{F}_i$ by open subtoposes $\mathcal{F}_i = \mathcal{F}_{o(U_i)}$ be given such that the induced geometric morphisms $f_i : \mathcal{E}_i \rightarrow \mathcal{F}_i$, where $\mathcal{E}_i = \mathcal{E}_{o(f^*U_i)}$, are all equivalences.*

$$\begin{array}{ccc} \mathcal{E}_i & \xrightarrow[\cong]{f_i} & \mathcal{F}_i \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{E} & \xrightarrow{f} & \mathcal{F} \end{array}$$

Then f is an equivalence.

Proof. The fact that the \mathcal{F}_i cover \mathcal{F} means $1_{\mathcal{F}} = \bigvee_{i \in I} U_i$ and, equivalently, the sheafification functors $a_{U_i} : \mathcal{F} \rightarrow \mathcal{F}_i$ are jointly conservative. Setting $V_i := f^*U_i$ we similarly have $1_{\mathcal{E}} = \bigvee_{i \in I} V_i$ since f^* preserves colimits. We will show that the unit η_f and counit ε_f of the adjunction $f^* \dashv f_*$ are isomorphisms by showing that $a_{U_i}\eta_f$ and $a_{V_i}\varepsilon_f$ are isomorphisms for all $i \in I$. More precisely, if η_{f_i} and ε_{f_i} are the unit and counit of $f_i^* \dashv f_{i*}$ we will show that $a_{U_i}\eta_f$ and $\eta_{f_i}a_{U_i}$ only differ by an isomorphism of their codomains $a_{U_i}f_*f^* \cong f_{i*}f_i^*a_{U_i}$, and similarly $a_{V_i}\varepsilon_f$ and $\varepsilon_{f_i}a_{V_i}$ only differ by an isomorphism of their domains $a_{V_i}f^*f_* \cong f_i^*f_{i*}a_{V_i}$.

Recall that $a_U : \mathcal{F} \rightarrow \mathcal{F}_{o(U)}$ has two fully faithful adjoints $j_U \dashv a_U \dashv i_U$, where $j_U(a_U(X)) = X \times U$ and $i_U(a_U(X)) = X^U$. Thus we have the following functors and adjunctions

$$\begin{array}{ccc} \mathcal{E}_{o(V)} & & \mathcal{F}_{o(U)} \\ j_V \downarrow \dashv \uparrow \downarrow i_V & & j_U \downarrow \dashv \uparrow \downarrow i_U \\ \mathcal{E} & \xleftarrow[f_*]{f^*} & \mathcal{F} \end{array}$$

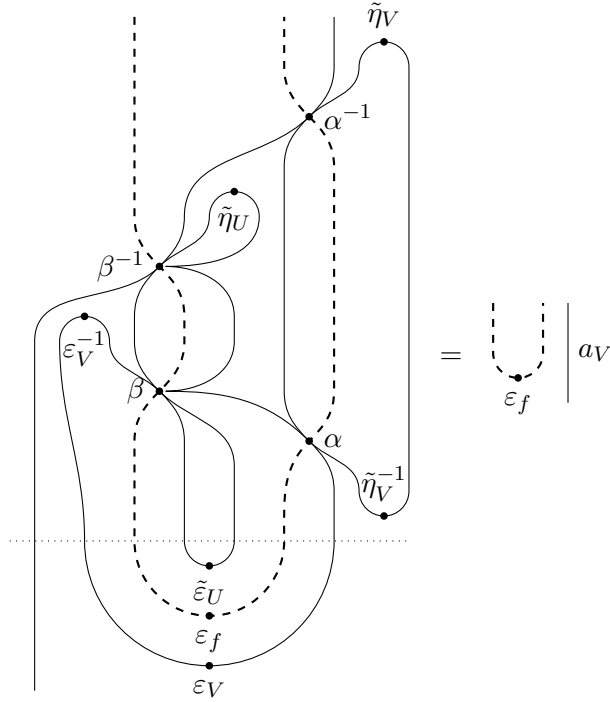
where $\varepsilon_U : a_U i_U \rightarrow \text{Id}_{\mathcal{F}_{o(U)}}$, $\tilde{\eta}_U : \text{Id}_{\mathcal{F}_{o(U)}} \rightarrow a_U j_U$, $\varepsilon_V : a_V i_V \rightarrow \text{Id}_{\mathcal{E}_{o(V)}}$ and $\tilde{\eta}_V : \text{Id}_{\mathcal{E}_{o(V)}} \rightarrow a_V j_V$ are isomorphisms. The geometric morphism $\mathcal{E}_{o(V)} \rightarrow \mathcal{F}_{o(U)}$ is given by $a_V f^* j_U \dashv a_U f_* i_V$. Our assumption therefore means that the following two natural transformations (denoted as string diagrams) are isomorphisms.

The only additional ingredient we need is a certain compatibility of f^* and f_* with the adjunctions $j_U \dashv a_U$ and $j_V \dashv a_V$. Since $f^*(X \times U) = f^*X \times V$ and $(\tilde{\varepsilon}_U)_X : X \times U \rightarrow X$ and $(\tilde{\varepsilon}_V)_Y : Y \times V \rightarrow Y$ are the projections on the first factor, we have a natural

isomorphism

For f_* the situation is slightly different as $f_*(V) \neq U$, but we have an arrow $(\eta_f)_U : U \rightarrow f_*(V)$ which becomes an isomorphism after applying a_U (since $f_*(V) \times U = U$). Thus we have an isomorphism

Now we can piece together isomorphisms reducing to $a_U \eta_f$ and $a_V \varepsilon_f$ respectively. (Start by introducing a new squiggle involving $\tilde{\varepsilon}_V$ and $\tilde{\eta}_V$ below $\tilde{\eta}_V^{-1}$ in the first diagram, and a new squiggle involving η_V and ε_V above ε_V^{-1} in the second diagram.)



□

Remark 3.5.2. Lemma 3.5.1 does not generalize to covers by arbitrary subtoposes, if cover just means that the join of these subtoposes is \mathcal{F} . For example, the intersection of any open subtopos $\mathcal{F}_{o(U)}$ with its closed complement $\mathcal{F}_{c(U)}$ is empty, so the geometric morphism

$$\mathcal{F}_{o(U)} \amalg \mathcal{F}_{c(U)} \rightarrow \mathcal{F}$$

from their disjoint union (which is given by the product of categories) becomes an equivalence when pulled back to either $\mathcal{F}_{o(U)}$ or $\mathcal{F}_{c(U)}$.

Corollary 3.5.3. *Let $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ be a model and let $(\phi_i)_{i \in I}$ be closed formulas such that \mathbb{T} proves $\bigvee_{i \in I} \phi_i$. If $M|_{\mathcal{E}_i}$ is a universal model of $\mathbb{T} + \phi_i$ for every $i \in I$, where $\mathcal{E}_i := \mathcal{E}_{o([\phi_i]_M)}$, then M was already a universal model of \mathbb{T} .*

Proof. Fixing a classifying topos $\text{Set}[\mathbb{T}]$ with universal model $M_{\mathbb{T}} \in \mathbb{T}\text{-mod}(\text{Set}[\mathbb{T}])$, we have a geometric morphism $f : \mathcal{E} \rightarrow \text{Set}[\mathbb{T}]$ with $f^*M_{\mathbb{T}} \cong M$. Since $\bigvee_{i \in I} \phi_i$ is provable, $U_i := [[\phi_i]]_{M_{\mathbb{T}}}$ define an open cover of $\text{Set}[\mathbb{T}]$ with $M_{\mathbb{T}}|_{\text{Set}[\mathbb{T}]_{o(U_i)}}$ a universal model of $\mathbb{T} + \phi_i$ and $f^*U_i \cong [[\phi_i]]_M$. The induced geometric morphisms $f_i : \mathcal{E}_i \rightarrow \text{Set}[\mathbb{T}]_{o(U_i)}$ are equivalences since $f_i^*(M|_{\text{Set}[\mathbb{T}]_{o(U_i)}}) \cong M|_{\mathcal{E}_i}$ is also a universal model of $\mathbb{T} + \phi_i$ by assumption. So Lemma 3.5.1 tells us that f is an equivalence, that is, $f^*M_{\mathbb{T}} \cong M$ is a universal \mathbb{T} -model. □

We can now give our construction of a syntactic presentation of a topos \mathcal{E} covered by open subtoposes \mathcal{E}_i , $i \in I$, from syntactic presentations of the \mathcal{E}_i and of the intersections

$\mathcal{E}_S := \bigcap_{i \in S} \mathcal{E}_i$ for $S \subseteq I$ finite and nonempty. The notation

$$\Delta(I) := \{ S \subseteq I \mid 1 \leq |S| < \infty \}$$

(thinking of simplices with vertices drawn from I) for the partial order indexing the required system of presentations will be useful. Note that we can write $S' \in \Delta(S)$ instead of $S' \leq S$ or $S' \subseteq S$ in this partial order.

In order to give an elegant description of the resulting theory, the open subtoposes \mathcal{E}_i will be assumed to be given by the interpretations of formulas ϕ_i of some base theory \mathbb{T}_0 in a (not necessarily universal) base model $M_0 \in \mathbb{T}_0\text{-mod}(\mathcal{E})$. This requirement can always be met, starting from an arbitrary base theory (e.g. the empty one), by adding proposition symbols p_i and the axiom $\bigvee_{i \in I} p_i$ to it, and then, since the p_i necessarily also show up in the presentation of each \mathcal{E}_j , add axioms to these presentations defining the p_i by already available formulas for the open subtoposes $\mathcal{E}_i \cap \mathcal{E}_j \subseteq \mathcal{E}_j$.

Theorem 3.5.4. *Let \mathcal{E} be a topos, \mathbb{T}_0 a geometric theory with closed formulas ϕ_i such that $\bigvee_{i \in I} \phi_i$ is provable, and $M_0 \in \mathbb{T}_0\text{-mod}(\mathcal{E})$ a model. For $S \in \Delta(I)$, set $\phi_S := \bigwedge_{i \in S} \phi_i$ and $\mathcal{E}_S := \mathcal{E}_{o(\llbracket \phi_S \rrbracket_{M_0})}$. Let a system of presentations $(\mathbb{E}_S, E_S)_{S \in \Delta(I)}$ of the \mathcal{E}_S over M_0 be given. Then \mathcal{E} classifies the theory*

$$\mathbb{T} := \mathbb{T}_0 + \sum_{S \in \Delta(I)} \mathbb{E}_S / \phi_S$$

with universal model

$$M := M_0 + \sum_{S \in \Delta(I)} E_S / \phi_S.$$

Proof. By Corollary 3.5.3, we only have to show that $M|_{\mathcal{E}_{\{i\}}}$ is a universal model of $\mathbb{T} + \phi_i$ for every $i \in I$. So fix i . We know that $M_0|_{\mathcal{E}_{\{i\}}} + E_{\{i\}}$, which lives in the same topos as $M|_{\mathcal{E}_{\{i\}}}$, is a universal model of $\mathbb{T}_0 + \mathbb{E}_{\{i\}}$ (in particular, $\mathbb{T}_0 + \mathbb{E}_{\{i\}}$ proves ϕ_i), and fortunately, $\mathbb{T} + \phi_i$ can be written as

$$\mathbb{T} + \phi_i = \mathbb{T}_0 + \mathbb{E}_{\{i\}} + \sum_{S \in \Delta_i(I)} (\mathbb{E}_{S \setminus \{i\}} + \mathbb{E}_S) / \phi_S,$$

where $\Delta_i(I) := \{ S \cup \{i\} \mid S \in \Delta(I \setminus \{i\}) \} = \{ S \in \Delta(I) \mid \{i\} \subsetneq S \}$. Also, the $(\mathbb{T}_0 + \mathbb{E}_{\{i\}})$ -part of $M|_{\mathcal{E}_{\{i\}}}$ is $M_0|_{\mathcal{E}_{\{i\}}} + E_{\{i\}}$. Thus, by Lemma 2.3.4, we are reduced to showing that the remaining extension $\sum_{S \in \Delta_i(I)} (\mathbb{E}_{S \setminus \{i\}} + \mathbb{E}_S) / \phi_S$ is an equivalence extension, which, by Lemma 3.3.3 and Lemma 3.2.4, we may do by showing that for every $S \in \Delta_i(I)$, the extension

$$\mathbb{E}_S^i := \mathbb{E}_{S \setminus \{i\}} + \mathbb{E}_S$$

is an equivalence extension of

$$\mathbb{T}_0 + \mathbb{E}_{\{i\}} + \phi_S + \sum_{S' \in \Delta_i(S), S' \neq S} \mathbb{E}_{S'}^i.$$

For this, observe that $M_0|_{\mathcal{E}_S} + E_{\{i\}}|_{\mathcal{E}_S}$ is a universal model in \mathcal{E}_S of $\mathbb{T}_0 + \mathbb{E}_{\{i\}} + \phi_S$, which we now treat as a base theory. And if we set

$$E_{S'}^i := E_{S' \setminus \{i\}}|_{\mathcal{E}_S} + E_{S'}|_{\mathcal{E}_S},$$

for every $S' \in \Delta(S)$, this in fact constitutes a system of presentations of the topos \mathcal{E}_S , since by assumption, every

$$M_0|_{\mathcal{E}_{S'}} + \sum_{S'' \in \Delta(S')} E_{S''}|_{\mathcal{E}_{S'}}$$

is a universal model, which can be restricted to a universal model in \mathcal{E}_S while adding ϕ_S to the theory. Then Corollary 3.3.5 allows us to conclude that in particular the topmost theory extension \mathbb{E}_S^i of this system is an equivalence extension, as needed. \square

Let us also show that a system of presentations as required in Theorem 3.5.4 always exists, starting from presentations of the \mathcal{E}_i (which exist simply by Theorem 2.2.2). In particular, it becomes clear that presentations are only needed for two- and threefold intersections of the open subtoposes \mathcal{E}_i , not for arbitrary finite intersections.

Proposition 3.5.5. *Let $M_0 \in \mathbb{T}_0\text{-mod}(\mathcal{E})$, let $\phi_i, i \in I$ be closed formulas of \mathbb{T}_0 and let ϕ_S, \mathcal{E}_S for $S \in \Delta(I)$ be as above. Then any family of presentations $(\mathbb{E}_{\{i\}}, E_{\{i\}})$ of the $\mathcal{E}_{\{i\}}$ over (\mathbb{T}_0, M_0) can be extended layer-wise to a system of presentations $(\mathbb{E}_S, E_S)_{S \in \Delta(I)}$ of the \mathcal{E}_S , where \mathbb{E}_S can be chosen localic for $|S| = 2$, a quotient for $|S| = 3$ and empty for $|S| \geq 4$. If the $\mathcal{E}_{\{i\}}$ are localic, then \mathbb{E}_S can even be chosen a quotient for $|S| = 2$ and empty for $|S| \geq 3$.*

Proof. Let $S \in \Delta(I)$ with $|S| \geq 2$ and let $\mathbb{E}_{S'}, E_{S'}$ be already defined for $S' \in \Delta(S), S' \neq S$. Fixing any $i \in S$ and setting $\tilde{S} := S \setminus \{i\}$, we regroup these extensions as

$$\mathbb{E}_{S'}^i := \mathbb{E}_{S'} + \mathbb{E}_{S' \cup \{i\}}$$

and include ϕ_S from the beginning to obtain a system of presentations $(\mathbb{E}_{S'}^i, E_{S'}^i|_{\mathcal{E}_S})$, $S' \in \Delta(\tilde{S}) \setminus \{\tilde{S}\}$ over $\mathbb{T}_0 + \mathbb{E}_{\{i\}} + \phi_S$ and $M_0|_{\mathcal{E}_S} + E_{\{i\}}|_{\mathcal{E}_S}$, all presenting \mathcal{E}_S . By Corollary 3.3.5, the sum of this system, which is

$$\mathbb{T}_0 + \phi_S + \sum_{S' \in \Delta(S) \setminus \{\tilde{S}, S\}} \mathbb{E}_{S'},$$

also presents \mathcal{E}_S . We can therefore apply Lemma 2.3.8 to revert the left-over extension $\mathbb{E}_{\tilde{S}}$ (with accompanying model extension $E_{\tilde{S}}$), obtaining \mathbb{E}_S and E_S such that

$$M_0|_{\mathcal{E}_S} + \sum_{S' \in \Delta(S)} E_{S'}|_{\mathcal{E}_S} \in \left(\mathbb{T}_0 + \sum_{S' \in \Delta(S)} \mathbb{E}_{S'} \right)\text{-mod}(\mathcal{E}_S)$$

is universal. (We can drop ϕ_S , since we definitely include all $\mathbb{E}_{\{j\}}$, $j \in S$ now.) And indeed \mathbb{E}_S is localic, and even a quotient (respectively empty) if $\mathbb{E}_{\tilde{S}}$ is localic (respectively a quotient). \square

3.6 The localic case

Proposition 3.5.5 suggests that a more concrete formulation of Theorem 3.5.4 might be feasible if the open subtoposes \mathcal{E}_i can be presented by localic extensions of an appropriate common base theory. This is the case in our applications to algebro-geometric toposes. We don't assume the subterminal objects U_i to be expressible by formulas of the base theory here. The theory extension adding a proposition symbol p will be denoted $\langle p \rangle$.

Corollary 3.6.1. *Let \mathcal{E} be a topos covered by open subtoposes $\mathcal{E}_i = \mathcal{E}_{o(U_i)}$, let $M_0 \in \mathbb{T}_0\text{-mod}(\mathcal{E})$ and let $E_i \in \mathbb{E}_i\text{-mod}(M_0|_{\mathcal{E}_i})$ be model extensions along localic extensions \mathbb{E}_i of \mathbb{T}_0 such that*

$$M_0|_{\mathcal{E}_i} + E_i \in (\mathbb{T}_0 + \mathbb{E}_i)\text{-mod}(\mathcal{E}_i)$$

is universal for every $i \in I$. Then:

- (i) For every i , there are closed formulas $\phi_{i,j}$ of $\mathbb{T}_0 + \mathbb{E}_i$ presenting the open subtoposes $\mathcal{E}_i \cap \mathcal{E}_j \subseteq \mathcal{E}_i$, and for every $i \neq j$, there is a diagonal quotient extension $\mathbb{Q}_{\{i,j\}}$ for $\mathbb{E}_i + \phi_{i,j}$ and $\mathbb{E}_j + \phi_{j,i}$ over \mathbb{T}_0 (see Definition 2.3.10) consisting of axioms fulfilled by $M_0|_{\mathcal{E}_i \cap \mathcal{E}_j} + E_1|_{\mathcal{E}_i \cap \mathcal{E}_j} + E_2|_{\mathcal{E}_i \cap \mathcal{E}_j}$.

- (ii) For any such $\phi_{i,j}$ and $\mathbb{Q}_{\{i,j\}}$, the topos \mathcal{E} classifies the theory

$$\begin{aligned} \mathbb{T} := & \mathbb{T}_0 + \langle p_i \rangle_{i \in I} + \left(\bigvee_{i \in I} p_i \right) + (\mathbb{E}_i/p_i)_{i \in I} \\ & + \left((p_j \dashv\vdash \phi_{i,j})/p_i \right)_{i \neq j \in I} + \left(\mathbb{Q}_{\{i,j\}}/(p_i \wedge p_j) \right)_{i \neq j \in I} \end{aligned}$$

with universal model

$$M := M_0 + ([p_i] := U_i)_{i \in I} + (E_i/p_i)_{i \in I}.$$

Note that finding appropriate quotient extensions $\mathbb{Q}_{\{i,j\}}$ just means stating enough properties of $M_0|_{\mathcal{E}_i \cap \mathcal{E}_j} + E_1|_{\mathcal{E}_i \cap \mathcal{E}_j} + E_2|_{\mathcal{E}_i \cap \mathcal{E}_j}$ as axioms to make the relation symbols of \mathbb{E}_2 definable by formulas of $\mathbb{T}_0 + \mathbb{E}_1$ and vice versa, see Remark 2.3.6.

Proof. The existence of the $\phi_{i,j}$ is clear. For $\mathbb{Q}_{\{i,j\}}$, observe that

$$\mathbb{T}_0 + \mathbb{E}_i + \phi_{i,j} \quad \text{and} \quad \mathbb{T}_0 + \mathbb{E}_j + \phi_{j,i}$$

both present the topos $\mathcal{E}_i \cap \mathcal{E}_j$, with universal models agreeing in the \mathbb{T}_0 -part. Thus, we obtain $\mathbb{Q}_{\{i,j\}}$ from Corollary 2.3.9.

Now set

$$\tilde{\mathbb{T}}_0 := \mathbb{T}_0 + \langle p_i \rangle_{i \in I} + \left(\bigvee_{i \in I} p_i \right), \quad \tilde{\mathbb{E}}_{\{i\}} := \mathbb{E}_i + p_i + (p_j \dashv\vdash \phi_{i,j})_{j \neq i}, \quad \tilde{\mathbb{E}}_{\{i,j\}} := \mathbb{Q}_{\{i,j\}}.$$

Then $\tilde{M}_0 := M_0 + ([p_i] := U_i)_{i \in I}$ is a model of $\tilde{\mathbb{T}}_0$ in \mathcal{E} , and $\tilde{M}_0|_{\mathcal{E}_i} + E_i$ is a universal model of $\tilde{\mathbb{T}}_0 + \tilde{\mathbb{E}}_{\{i\}}$, since this only differs from $\mathbb{T}_0 + \mathbb{E}_i$ by an extension by definitions, and similarly for $\tilde{\mathbb{E}}_{\{i,j\}}$. By Proposition 3.5.5, this system of presentations can be extended by setting $\tilde{\mathbb{E}}_S = \emptyset$ for all $S \in \Delta(I)$ with $|S| \geq 3$. Then we can apply Theorem 3.5.4 and obtain the theory \mathbb{T} with universal model M as stated. (We only split up $\tilde{\mathbb{E}}_{\{i\}}/p_i$ and dropped the trivial extension p_i/p_i .) \square

3.7 Application to Zariski toposes

We now apply our results to deduce a syntactic presentation for the big Zariski topos of an arbitrary scheme, which can be viewed as glued from big Zariski toposes of affine schemes. We first recall the definition, and the classifying property in the affine case.

The *big Zariski site* of a scheme S is the category Sch/S of schemes over S , equipped with the Zariski topology J_{Zar} in which a sieve on an object $T \rightarrow S$ is covering if and only if there is an open cover $T = \bigcup_i T_i$ of the scheme T such that the sieve contains the open embeddings

$$\begin{array}{ccc} T_i & \hookrightarrow & T \\ & \searrow & \swarrow \\ & S & \end{array}$$

The site defined in this way does not admit any (small) dense set of objects, so one has to restrict the class of objects in an appropriate way before the sheaf topos can be formed. (The resulting topos is still called a *big Zariski topos*, unless the site only contains open subschemes of S .) We choose the subcategory $(\text{Sch}/S)_{\text{lofp}}$ of schemes locally of finite presentation over S , yielding the *big Zariski topos*

$$S_{\text{Zarfp}} := \text{Sh}((\text{Sch}/S)_{\text{lofp}}, J_{\text{Zar}}).$$

The motivation for this particular class of objects is that in the affine case $S = \text{Spec } K$ (where K is any commutative unitary ring), the big Zariski topos $(\text{Spec } K)_{\text{Zarfp}}$ is the classifying topos of the theory of local K -algebras.

Definition 3.7.1.

- (i) The (algebraic) theory of rings will be denoted Ring . It has one sort A , function symbols $0, 1 : A$, $- : A \rightarrow A$, $+, \cdot : A \times A \rightarrow A$ and the usual axioms of a commutative unitary ring.
- (ii) In the theory Ring (or any extension of it), we use the abbreviation

$$\text{inv}(x) \quad := \quad \exists \bar{x} : A. (x\bar{x} = 1).$$

- (iii) We can require a ring to be local by the quotient extension

$$(\text{loc}) := \left\{ \exists \vec{y} : A^n. \left(\sum_{i=1}^n x_i y_i = 1 \right) \vdash_{\vec{x}:A^n} \bigvee_{i \in \{1, \dots, n\}} \text{inv}(x_i) \quad \mid \quad n \in \mathbb{N} \right\}.$$

(An alternative, syntactically equivalent axiomatization is given by the two sequents $0 = 1 \vdash_{\square} \perp$ and $\top \vdash_{x:A} \text{inv}(x) \vee \text{inv}(1 - x)$.)

- (iv) For a ring K , the extension $K\text{-AlgStr}$ of the theory Ring consists of constant symbols $c_\lambda : A$ for all $\lambda \in K$ and the axioms (for all $\lambda, \mu \in K$ as needed)

$$c_0 = 0, \quad c_\lambda + c_\mu = c_{\lambda+\mu}, \quad c_1 = 1, \quad c_\lambda c_\mu = c_{\lambda\mu}.$$

(v) The theory of K -algebras is

$$K\text{-Alg} := \text{Ring} + K\text{-AlgStr}.$$

A quick sketch of a proof that $(\text{Spec } K)_{\text{Zar}_{\text{fp}}}$ classifies $K\text{-Alg} + (\text{loc})$, using notions from Section 4, goes as follows. Since the theory $K\text{-Alg}$ is algebraic, a presheaf site for its classifying topos is given by the opposite of the category of finitely presented K -algebras. But this is equivalent to the full subcategory on the affine objects of $(\text{Sch}/\text{Spec } K)_{\text{lofp}}$, which is a dense subcategory with respect to J_{Zar} , and therefore, equipped with the topology induced by J_{Zar} , an alternative site for the same topos by the Comparison Lemma. Finally, the subtopos defined by this Zariski topology on the finitely presented K -algebras is the subtopos presented by the quotient extension (loc) .

The universal model of $K\text{-Alg} + (\text{loc})$ in $(\text{Spec } K)_{\text{Zar}_{\text{fp}}}$ is the structure sheaf

$$\mathcal{O} = \mathcal{O}_{S_{\text{Zar}_{\text{fp}}}} : (\text{Sch}/S)_{\text{lofp}}^{\text{op}} \rightarrow \text{Set}, \quad T \mapsto \mathcal{O}_T(T).$$

Remark 3.7.2. The theory extension $K\text{-AlgStr}$ (and therefore the theory $K\text{-Alg}$) can be formulated more economically, using a presentation of the ring K by generators and relations. Namely, if $K = \mathbb{Z}[X_i]/(r_j)$, where i and j run over any two index sets, it suffices to add constant symbols $c_i : A$ and axioms $r_j(c_i) = 0$ to the theory of rings, where $r_j(c_i)$ is to be interpreted as a closed term, using the available ring structure. The theory $K\text{-Alg}$ from Definition 3.7.1 can then be obtained by an extension by definitions. So for example, the topos $(\text{Spec } \mathbb{Z}[X])_{\text{Zar}_{\text{fp}}}$ classifies the theory of local rings with one distinguished element.

The big Zariski topos is functorial in the scheme S , in that a morphism $S' \rightarrow S$ of schemes induces a geometric morphism $S'_{\text{Zar}_{\text{fp}}} \rightarrow S_{\text{Zar}_{\text{fp}}}$, see [7, Tag 0210]. But we only need the special case where S' is an open subscheme of S here. In this case, observe that S' can be regarded as a subterminal object of the site $(\text{Sch}/S)_{\text{lofp}}$ and, since the Zariski topology J_{Zar} is subcanonical, determines a subterminal sheaf

$$U_{S'} := \text{Hom}_S(-, S') : (\text{Sch}/S)_{\text{lofp}}^{\text{op}} \rightarrow \text{Set}.$$

Lemma 3.7.3.

(i) *The mapping*

$$S' \mapsto (S_{\text{Zar}_{\text{fp}}})_{o(U_{S'})}$$

from open subschemes of S to open subtoposes of $S_{\text{Zar}_{\text{fp}}}$ is monotone and preserves finite intersections and arbitrary unions.

(ii) *For any open subscheme S' of S , the open subtopos $(S_{\text{Zar}_{\text{fp}}})_{o(U_{S'})}$ is equivalent to $S'_{\text{Zar}_{\text{fp}}}$. The inverse image ι^* and the further left adjoint $\iota_!$ of the open embedding $\iota : S'_{\text{Zar}_{\text{fp}}} \hookrightarrow S_{\text{Zar}_{\text{fp}}}$ are given by*

$$(\iota^* F)(T') = F(T') \quad (T' \in (\text{Sch}/S')_{\text{lofp}})$$

and

$$(\iota_! F')(T) = \begin{cases} F'(T), & \text{if } T \rightarrow S \text{ factors through } S' \\ \emptyset, & \text{otherwise} \end{cases} \quad (T \in (\text{Sch}/S)_{\text{lofp}}).$$

(iii) If $S = \text{Spec } K$ is affine (and therefore S_{Zarfp} classifies $K\text{-Alg} + (\text{loc})$) and $f \in K$, then the open subtopos $(S_{\text{Zarfp}})_{o(U_{D(f)})}$ corresponding to the standard open $D(f) \subseteq S$ is presented by the closed geometric formula

$$\text{inv}(c_f) \quad := \quad \exists x : A. (xc_f = 1).$$

Proof. (i) We can identify open subtoposes with subterminal sheaves $F \subseteq 1_{S_{\text{Zarfp}}}$, which can in turn be thought of as certain classes of objects of $(\text{Sch}/S)_{\text{lofp}}$, where $(S_{\text{Zarfp}})_{o(U_{S'})}$ corresponds to the class of all scheme morphisms $T \rightarrow S$ (locally of finite presentation) that factor through S' . Monotonicity is then clear, as well as $U_S = 1_{S_{\text{Zarfp}}}$. For binary intersections, we have to check that $T \rightarrow S$ factors through $S'_1 \cap S'_2$ if and only if it factors through S'_1 and S'_2 . But this is also clear. For unions, let $(S'_i)_{i \in I}$ be some family of open subschemes and let $F \subseteq 1_{S_{\text{Zarfp}}}$ be given with $U_{S'_i} \leq F$ for all i , that is, $|F(S'_i)| = 1$. Because $S' := \bigcup_{i \in I} S'_i$ is covered by the S'_i with respect to J_{Zar} , this implies $|F(S')| = 1$, so $U_{S'} \leq F$, as needed.

(ii) The underlying category of the open subtopos $(S_{\text{Zarfp}})_{o(U_{S'})}$ is equivalent to

$$\text{Sh}((\text{Sch}/S)_{\text{lofp}}, J_{\text{Zar}}) / \text{Hom}_S(-, S'),$$

the category of all those sheaves F on $(\text{Sch}/S)_{\text{lofp}}$ for which $F(T) \neq \emptyset$ implies that $T \rightarrow S$ factors through S' . The category of presheaves on $(\text{Sch}/S)_{\text{lofp}}$ with this property is clearly equivalent to $\text{PSh}((\text{Sch}/S')_{\text{lofp}})$, and one can check that the sheaf conditions with respect to the two Zariski topologies are then equivalent. Then we have

$$(\iota^* F)(T') = (F \times U_{S'})(T') = F(T')$$

for $T' \rightarrow S'$, and the further left adjoint $\iota_!$ is the forgetful functor $S_{\text{Zarfp}}/U_{S'} \rightarrow S_{\text{Zarfp}}$, as stated.

(iii) We know that S_{Zarfp} classifies $K\text{-Alg} + (\text{loc})$ with universal model $\mathcal{O} = \mathcal{O}_{S_{\text{Zarfp}}}$, and $(D(f))_{\text{Zarfp}} = (\text{Spec } K_f)_{\text{Zarfp}}$ classifies $K_f\text{-Alg} + (\text{loc})$, which is equivalent to $K\text{-Alg} + (\text{loc}) + (\text{inv}(c_f))$, with universal model $\mathcal{O}' = \mathcal{O}_{(D(f))_{\text{Zarfp}}}$. So to show that the embedding ι is presented by the quotient extension (loc) , all we have to check is that $\iota^* \mathcal{O} \cong \mathcal{O}'$ as models of $K\text{-Alg}$. And this follows from the description of ι^* in (ii). □

Remark 3.7.4. Not all open subtoposes of S_{Zarfp} correspond to open subschemes of S in this way. For example, $\text{Hom}_S(-, S')$ is also a subterminal object if S' is a *closed* subscheme of S .

Before we formulate the theory classified by $S_{\text{Zar}_{\text{fp}}}$ in generality, we discuss the example of the projective line.

Proposition 3.7.5. *The big Zariski topos $(\mathbb{P}_K^1)_{\text{Zar}_{\text{fp}}}$ of the projective line over a ring K classifies the theory $\mathbb{T}_{\mathbb{P}_K^1}$, which is $K\text{-Alg} + (\text{loc})$ expanded by two relation symbols $\tilde{c}_1, \tilde{c}_2 \subseteq A$ and certain axioms as follows.*

$$\begin{aligned} \mathbb{T}_{\mathbb{P}_K^1} := & K\text{-Alg} + (\text{loc}) + \langle \tilde{c}_1, \tilde{c}_2 \subseteq A \rangle \\ & + \left(x \in \tilde{c}_i \wedge x' \in \tilde{c}_i \vdash_{x,x'} x = x' \right)_{i=1,2} \\ & + \left(\top \vdash_{\emptyset} (\exists x. x \in \tilde{c}_1) \vee (\exists x. x \in \tilde{c}_2) \right) \\ & + \left(x_1 \in \tilde{c}_1 \wedge x_2 \in \tilde{c}_2 \vdash_{x_1,x_2} x_1 x_2 = 1 \right) \\ & + \left(x \in \tilde{c}_i \wedge \text{inv}(x) \vdash_x \exists y. y \in \tilde{c}_{i'} \right)_{(i,i')=(1,2),(2,1)} \end{aligned}$$

The relation symbols $\tilde{c}_1, \tilde{c}_2 \subseteq A$ should be understood as “partial constants” $c_1, c_2 : A$, with the following properties. At least one of them is defined, and if one is defined then the other is its inverse, which may or may not be defined.

Proof. An affine open cover of the scheme \mathbb{P}_K^1 is given by two copies of the affine line \mathbb{A}_K^1 , we write

$$\mathbb{P}_K^1 = \text{Spec } K[X_1] \cup \text{Spec } K[X_2],$$

and their intersection is

$$\text{Spec } K[X_1] \cap \text{Spec } K[X_2] = \text{Spec } K[X_1, X_2]/(X_1 X_2 - 1),$$

with the open inclusions corresponding to the K -algebra maps suggested by the notation. In other words, as an open subscheme of each $\text{Spec } K[X_i]$, the overlap is the standard open $D(X_i)$, and the identification corresponds to the isomorphism of rings is

$$\varphi : K[X_1]_{X_1} \rightarrow K[X_2]_{X_2}, \quad X_1 \mapsto X_2^{-1}.$$

We want to apply Corollary 3.6.1 to the open cover of toposes

$$(\mathbb{P}_K^1)_{\text{Zar}_{\text{fp}}} = \mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$$

induced by the above via Lemma 3.7.3. Consider the structure sheaf of $(\mathbb{P}_K^1)_{\text{Zar}_{\text{fp}}}$, which is a sheaf of K -algebras, as a model

$$M_0 := \mathcal{O} \in (K\text{-Alg})\text{-mod}(\mathcal{E}).$$

Then the $(K\text{-Alg})$ -models $M_0|_{\mathcal{E}_i}$ are the respective structure sheaves (by Lemma 3.7.3 (ii)), that is, they become universal models of $K\text{-Alg} + \langle c_i : A \rangle + (\text{loc})$ when the constant symbol c_i is interpreted as the global section of $M_0|_{\mathcal{E}_i} = \mathcal{O}_{(\text{Spec } K[X_i])_{\text{Zar}_{\text{fp}}}}$ corresponding to X_i . In particular, M_0 satisfies (loc) , so we can use as our base theory

$$\mathbb{T}_0 := K\text{-Alg} + (\text{loc}),$$

and we have presentations $(\mathbb{T}_0 + \mathbb{E}_i, M_0|_{\mathcal{E}_i} + E_i)$ of the \mathcal{E}_i with

$$\mathbb{E}_i := \langle c_i : A \rangle.$$

Using Lemma 3.7.3 (ii), the subtopos $\mathcal{E}_1 \cap \mathcal{E}_2$ is presented in \mathcal{E}_i by the closed geometric formula

$$\phi_i := \text{inv}(c_i).$$

It remains to find a diagonal quotient extension of $\mathbb{E}_1 + \phi_1$ and $\mathbb{E}_2 + \phi_2$ over \mathbb{T}_0 satisfied by $\mathcal{O}_{\mathcal{E}_1 \cap \mathcal{E}_2}$ (with the interpretations of c_i coming from $\mathcal{O}_{\mathcal{E}_i}$). Since $\mathcal{O}_{\mathcal{E}_1 \cap \mathcal{E}_2}$ is the universal local $K[X_1, X_2]/(X_1X_2 - 1)$ -algebra, we can take

$$\mathbb{Q}_{\{1,2\}} := \left(\top \vdash_{\square} c_1c_2 = 1 \right).$$

Now, to form conditional extensions like \mathbb{E}_i/p_i , we have to replace the constant symbols c_i by relation symbols $\tilde{c}_i \subseteq A$ with appropriate axioms, and rewrite the formulas in which they appear. (We need to “desugar” the syntax.) Thus we obtain

$$\begin{aligned} \mathbb{E}_i &= \langle \tilde{c}_i \subseteq A \rangle + \left(\top \vdash_{\square} \exists x. x \in \tilde{c}_i \right) + \left(x \in \tilde{c}_i \wedge x' \in \tilde{c}_i \vdash_{x,x'} x = x' \right), \\ \phi_i &= \exists x. (x \in \tilde{c}_i \wedge \text{inv}(x)), \\ \mathbb{Q}_{\{1,2\}} &= \exists x_1, x_2. (x_1 \in \tilde{c}_1 \wedge x_2 \in \tilde{c}_2 \wedge x_1x_2 = 1). \end{aligned}$$

Then Corollary 3.6.1 gives us a theory \mathbb{T} classified by $(\mathbb{P}_K^1)_{\text{Zar}_{\text{fp}}}$. This \mathbb{T} contains proposition symbols p_1, p_2 , which can however be eliminated since \mathbb{T} proves

$$p_i \dashv\vdash_{\square} \exists x. x \in \tilde{c}_i.$$

(This is always the case when the extensions \mathbb{E}_i contain at least one constant symbol.) Then one can check that the resulting theory is syntactically equivalent to $\mathbb{T}_{\mathbb{P}_K^1}$ as defined in the statement. \square

In the general case, we will of course have more than two affine schemes in the covering, but also, their intersections will not be given as single standard open subschemes. Recall, however, that the intersection $S_1 \cap S_2$ of two affine open subschemes of a scheme S can always be covered by open subschemes which are standard opens in both S_1 and S_2 [9, 5.3.1. Proposition].

Theorem 3.7.6. *Let $S = \bigcup_{i \in I} S_i$ be a scheme covered by affine open subschemes $S_i = \text{Spec } K_i$, and for every $i \neq i' \in I$, let an open cover*

$$S_i \cap S_{i'} = \bigcup_{j \in J_{\{i,i'\}}} S_{\{i,i'\}}^j$$

be given, such that

$$\text{Spec } K_i \supseteq D(f_{i,i'}^j) = S_{\{i,i'\}}^j = D(f_{i',i}^j) \subseteq \text{Spec } K_{i'},$$

with corresponding ring isomorphisms

$$\varphi_{i,i'}^j = (\varphi_{i',i}^j)^{-1} : (K_i)_{f_{i,i'}^j} \rightarrow (K_{i'})_{f_{i',i}^j}.$$

Then the big Zariski topos $S_{\text{Zar}_{\text{fp}}}$ classifies the theory

$$\begin{aligned} \mathbb{T}_S := & \text{Ring} + (\text{loc}) + \langle p_i \rangle_{i \in I} + \left(\bigvee_{i \in I} p_i \right) + ((K_i\text{-AlgStr})/p_i)_{i \in I} \\ & + \left(x \in \tilde{c}_{f_{i,i'}^j} \wedge \text{inv}(x) \vdash_x p_{i'} \right)_{i \neq i' \in I, j \in J_{\{i,i'\}}} \\ & + \left(p_i \wedge p_{i'} \vdash_{\square} \bigvee_{j \in J_{\{i,i'\}}} \exists x. (x \in \tilde{c}_{f_{i,i'}^j} \wedge \text{inv}(x)) \right)_{i \neq i' \in I, j \in J_{\{i,i'\}}} \\ & + \left(x \in \tilde{c}_{f_{i',i}^j} \wedge \text{inv}(x) \wedge y \in \tilde{c}_\lambda \wedge z \in \tilde{c}_{\lambda'} \vdash_{x,y,z} x^n y = z \right)_{i \neq i' \in I, j \in J_{\{i,i'\}}, \lambda \in K_i}, \end{aligned}$$

where in the last family of axioms, $\lambda' \in K_{i'}$ and $n \in \mathbb{N}$ are chosen for each $\lambda \in K_i$ such that $\varphi_{i,i'}^j(\lambda) = (f_{i',i}^j)^{-n} \lambda'$.

Proof. By Lemma 3.7.3, the topos $\mathcal{E} := S_{\text{Zar}_{\text{fp}}}$ is covered by the open subtoposes $\mathcal{E}_i := (S_i)_{\text{Zar}_{\text{fp}}}$. The restrictions $\mathcal{O}|_{\mathcal{E}_i}$ of the Ring-model $\mathcal{O} = \mathcal{O}_{S_{\text{Zar}_{\text{fp}}}}$ can be extended by K_i -algebra structures, resulting in universal models of $K_i\text{-Alg} + (\text{loc})$. So we can use the base theory

$$\mathbb{T}_0 := \text{Ring} + (\text{loc})$$

with the model

$$M_0 := \mathcal{O} \in (\text{Ring} + (\text{loc}))\text{-mod}(\mathcal{E}),$$

and have presentations of the \mathcal{E}_i with

$$\mathbb{E}_i := K_i\text{-AlgStr}.$$

The open subtopos $\mathcal{E}_i \cap \mathcal{E}_{i'} \subseteq \mathcal{E}_i$ is then presented by the closed geometric formula

$$\phi_{i,i'} := \bigvee_{j \in J_{\{i,i'\}}} \text{inv}(c_{f_{i,i'}^j}).$$

To find appropriate diagonal quotient extensions $\mathbb{Q}_{\{i,i'\}}$, consider the model

$$M_0|_{\mathcal{E}_i \cap \mathcal{E}_{i'}} = \mathcal{O}_{(S_i \cap S_{i'})_{\text{Zar}_{\text{fp}}}},$$

which carries both a K_i -algebra structure and a $K_{i'}$ -algebra structure. For every $j \in J_{\{i,i'\}}$, the further restriction to $(S_{\{i,i'\}}^j)_{\text{Zar}_{\text{fp}}}$ is the universal local $(K_i)_{f_{i,i'}^j}$ -algebra and at the same time the universal local $(K_{i'})_{f_{i',i}^j}$ -algebra, with the algebra structures coinciding via the isomorphism $\varphi_{i,i'}^j$. For every $j \in J_{\{i,i'\}}$ and every $\lambda \in K_i$, write

$$\varphi_{i,i'}^j(\lambda) = (f_{i',i}^j)^{-n} \lambda'$$

for some $\lambda' = \lambda'_{j,\lambda} \in K_{i'}$, $n = n_{j,\lambda} \in \mathbb{N}$. Then $\mathcal{O}_{(S_i \cap S_{i'})_{\text{Zar}_{\text{fp}}}}$ satisfies the axioms

$$\begin{aligned} \mathbb{Q}_{i,i'} &:= \left\{ \text{inv}(c_{f_{i',i}^j}) \vdash_{\square} (c_{f_{i',i}^j})^n c_{\lambda} = c_{\lambda'} \mid j \in J_{\{i,i'\}}, \lambda \in K_i \right\}, \\ \mathbb{Q}_{\{i,i'\}} &:= \mathbb{Q}_{i,i'} + \mathbb{Q}_{i',i}. \end{aligned}$$

Note that the sequents in $\mathbb{Q}_{i,i'}$ also make $\text{inv}(c_{f_{i',i}^j}) \vdash_{\square} \text{inv}(c_{f_{i,i'}^j})$ provable, because we can set $\lambda := f_{i,i'}^j$ and then λ' will be invertible as an element of $(K_{i'})_{f_{i',i}^j}$.

The quotient extension $\mathbb{Q}_{i,i'}$ clearly makes each c_{λ} definable in terms of the $K_{i'}$ -algebra structure if we assume one of the formulas $\text{inv}(c_{f_{i',i}^j})$. But we need c_{λ} to be definable by a single formula of $\mathbb{T}_0 + \mathbb{E}_{i'} + \phi_{i',i}$. Thus, consider

$$\phi_{c_{\lambda}}(x) := \bigvee_{j \in J_{\{i,i'\}}} \left(\text{inv}(c_{f_{i',i}^j}) \wedge (c_{f_{i',i}^j})^n x = c_{\lambda'} \right).$$

Then we see that $\mathbb{Q}_{i,i'}$ is exactly what we need, together with $\phi_{i',i}$, to prove

$$\phi_{c_{\lambda}}(x) \dashv\vdash_x x = c_{\lambda},$$

completing our argument why $\mathbb{Q}_{\{i,i'\}}$ is a diagonal quotient extension of $\mathbb{E}_i + \phi_{i,i'}$ and $\mathbb{E}_{i'} + \phi_{i',i}$ over \mathbb{T}_0 . (It is automatic that $\phi_{c_{\lambda}}(x)$ is provably functional as a formula of $\mathbb{T}_0 + \mathbb{E}_{i'} + \phi_{i',i}$, see Remark 2.3.6.)

Now Corollary 3.6.1 can finally be applied to the data \mathbb{T}_0 , \mathbb{E}_i , $\phi_{i,i'}$ and $\mathbb{Q}_{\{i,i'\}}$, considering all the constant symbols $c_{\lambda} : A$ to be “syntactic sugar” for relation symbols $\tilde{c}_{\lambda} \subseteq A$ together with appropriate axioms. The resulting theory \mathbb{T} is syntactically equivalent to \mathbb{T}_S as given in the statement. We have only simplified the axioms by dropping p_i in the antecedent whenever a formula of the form $x \in \tilde{c}_{\lambda}$ appears there too, for some $\lambda \in K_i$, since the equivalence

$$p_i \dashv\vdash_{\square} \exists x. x \in \tilde{c}_{\lambda}$$

is already contained in $(K_i\text{-AlgStr})/p_i$. \square

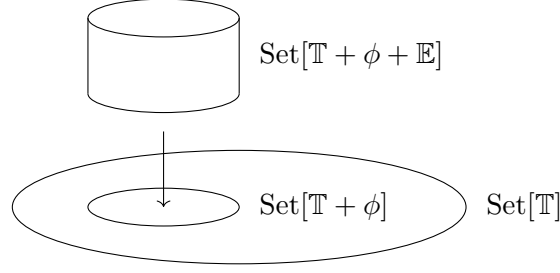
Remark 3.7.7. Theorem 3.7.6 can be regarded as a generalization of [2, Proposition 16.3], where the Set-based points of $S_{\text{Zar}_{\text{fp}}}$ are identified as the “local rings over S ”, that is, local rings A together with a scheme morphism $\text{Spec } A \rightarrow S$. One can check that this category is indeed equivalent to $\mathbb{T}_S\text{-mod}(\text{Set})$.

3.8 Conditional extensions and Artin gluing

We now consider the question what the classifying topos of a conditionally extended theory $\mathbb{T} + \mathbb{E}/\phi$ looks like, that is, we aim to describe $\text{Set}[\mathbb{T} + \mathbb{E}/\phi]$ in terms of the geometric morphisms

$$\text{Set}[\mathbb{T} + \phi + \mathbb{E}] \xrightarrow{\pi_{\mathbb{E}}} \text{Set}[\mathbb{T} + \phi] \xleftarrow{\pi_{\phi}} \text{Set}[\mathbb{T}],$$

where π_{ϕ} is an open embedding, but $\pi_{\mathbb{E}}$ can be an arbitrary geometric morphism (see Theorem 2.2.2). This situation can perhaps be visualized by the following picture.



In the special case where the base theory is simply the theory $\langle p \rangle$ of a proposition symbol p with no axioms, we can give an answer directly by analyzing the syntactic site of the resulting theory. Recall that the *Sierpiński cone* (or Freyd cover) $\text{scn}(\mathcal{E})$ of a topos \mathcal{E} is the topos with underlying category the comma category $(\text{Set} \downarrow \Gamma)$, where $\Gamma : \mathcal{E} \rightarrow \text{Set}$ is the global sections functor, and that it has a canonical subterminal object $U = (\emptyset, 1_{\mathcal{E}}, !)$, where $! : \emptyset \rightarrow \Gamma(1_{\mathcal{E}}) = \{*\}$, such that the corresponding open subtopos is $\text{scn}(\mathcal{E})_{o(U)} \simeq \mathcal{E}$, while its closed complement is just a point, $\text{scn}(\mathcal{E})_{c(U)} \simeq \text{Set}$. This corresponds to the fact that $\langle p \rangle + \mathbb{T}/p + p$ is equivalent to \mathbb{T} and $\langle p \rangle + \mathbb{T}/p + \neg p$ is equivalent to the empty theory (over the empty signature) classified by Set .

Proposition 3.8.1. *For any geometric theory \mathbb{T} , the theory $\langle p \rangle + \mathbb{T}/p$ is classified by the Sierpiński cone $\text{scn}(\text{Set}[\mathbb{T}])$.*

Proof. A construction of the Sierpiński cone on the level of sites proceeds as follows. Given a site (C, J) , we define C' by freely adjoining a terminal object to C , that is, $\text{Ob}(C') = \text{Ob}(C) \sqcup \{*\}$ and $|\text{Hom}(c, *)| = 1$, $\text{Hom}(*, c) = \emptyset$ for all $c \in \text{Ob}(C)$. The topology J' is given by simply leaving the covering sieves of any $c \in C$ (which are still sieves in C' , as there is no arrow $* \rightarrow c$) as they are and declaring only the maximal sieve on $*$ to be covering. The axioms of a Grothendieck topology are clearly satisfied. Then a sheaf F' on (C', J') is exactly a sheaf F on (C, J) together with a set $F'(*)$ and a map $F'(*) \rightarrow \Gamma(F)$, as required.

We claim that this is precisely the relationship between the syntactic sites $C_{\mathbb{T}}$ and $C_{\langle p \rangle + \mathbb{T}/p}$. First note that we have a full and faithful functor

$$C_{\langle p \rangle + \mathbb{T}/p+p} \rightarrow C_{\langle p \rangle + \mathbb{T}/p}, \quad \{\vec{x}. \phi\} \mapsto \{\vec{x}. \phi \wedge p\}$$

with essential image $C_{\langle p \rangle + \mathbb{T}/p}/\{\square. p\}$, the subcategory of all $\{\vec{x}. \phi\}$ with $\phi \vdash_{\vec{x}} p$ provable in $\langle p \rangle + \mathbb{T}/p$. (This is true for any closed formula of any geometric theory.) But of course $C_{\langle p \rangle + \mathbb{T}/p+p} \simeq C_{\mathbb{T}}$. And if an object $\{\vec{x}. \phi\} \in C_{\langle p \rangle + \mathbb{T}/p}$ does not lie over $\{\square. p\}$, we can show that the context \vec{x} must be empty (because $\langle p \rangle + \mathbb{T}/p$ contains the axiom $\top \vdash_x :_S p$ for any of its sorts) and, by induction over ϕ , that $\top \vdash_{\square} \phi$ is provable in $\langle p \rangle + \mathbb{T}/p$. So the only missing object (up to isomorphism) is the terminal object $\{\square. \top\}$, and this terminal object is strict, since $\top \vdash_{\square} p$ is not provable in $\langle p \rangle + \mathbb{T}/p$. It is easy to verify that the Grothendieck topology on $C_{\langle p \rangle + \mathbb{T}/p}$ is also the one coming from $C_{\mathbb{T}}$ as above. \square

For the general case, recall from [5, Proposition 4.5.6] that any topos \mathcal{E} equipped with a subterminal object U can be reconstructed from the toposes $\mathcal{E}_{o(U)}$ and $\mathcal{E}_{c(U)}$ and the

left exact functor

$$F := j^* \circ i_* : \mathcal{E}_{o(U)} \rightarrow \mathcal{E}_{c(U)},$$

called the *fringe functor* of \mathcal{E} (with respect to U), where $i : \mathcal{E}_{o(U)} \rightarrow \mathcal{E}$ and $j : \mathcal{E}_{c(U)} \rightarrow \mathcal{E}$ are the inclusion geometric morphisms. Namely, \mathcal{E} is equivalent to the *Artin gluing* of F , defined, for any left exact functor $F : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ between toposes, as the comma category

$$\mathrm{Gl}(F) := (\mathcal{E}_2 \downarrow F),$$

equipped with the subterminal object $(\emptyset_{\mathcal{E}_2}, 1_{\mathcal{E}_1}, !)$, where $! : \emptyset_{\mathcal{E}_2} \rightarrow F(1_{\mathcal{E}_1}) = 1_{\mathcal{E}_2}$. In our situation, the topos $\mathrm{Set}[\mathbb{T} + \mathbb{E}/\phi]$ is therefore an Artin gluing of the open subtopos

$$\mathrm{Set}[\mathbb{T} + \mathbb{E}/\phi + \phi] = \mathrm{Set}[\mathbb{T} + \phi + \mathbb{E}]$$

and its closed complement

$$\mathrm{Set}[\mathbb{T} + \mathbb{E}/\phi + \neg\phi] = \mathrm{Set}[\mathbb{T} + \neg\phi + \mathbb{E}/\perp] \simeq \mathrm{Set}[\mathbb{T} + \neg\phi].$$

Conjecture 3.8.2. Let \mathbb{T} be a geometric theory, ϕ a closed formula of \mathbb{T} and \mathbb{E} an extension of $\mathbb{T} + \phi$. Then the theory $\mathbb{T} + \mathbb{E}/\phi$ is classified by the topos

$$\mathrm{Gl}\left(\mathrm{Set}[\mathbb{T} + \phi + \mathbb{E}] \xrightarrow{\pi_{\phi + \mathbb{E}*}} \mathrm{Set}[\mathbb{T}] \xrightarrow{\pi_{\neg\phi}^*} \mathrm{Set}[\mathbb{T} + \neg\phi]\right).$$

The functor $\pi_{\neg\phi}^* \circ \pi_{\phi + \mathbb{E}*}$ can also be understood as the composition $F \circ \pi_{\mathbb{E}*}$, where $F : \mathrm{Set}[\mathbb{T} + \phi] \rightarrow \mathrm{Set}[\mathbb{T} + \neg\phi]$ is the fringe functor of $\mathrm{Set}[\mathbb{T}]$, that is, $\mathrm{Gl}(F) \simeq \mathrm{Set}[\mathbb{T}]$. What the conjecture says, then, is that the topos $\mathrm{Set}[\mathbb{T} + \mathbb{E}/\phi]$ is the result of replacing the open subtopos $\mathrm{Set}[\mathbb{T} + \phi]$ of $\mathrm{Set}[\mathbb{T}]$ by $\mathrm{Set}[\mathbb{T} + \phi + \mathbb{E}]$ in a canonical way, using the given geometric morphism $\pi_{\mathbb{E}}$.

To prove it, we would have to show that the given left exact functor is isomorphic to the fringe functor of $\mathrm{Set}[\mathbb{T} + \mathbb{E}/\phi]$. One attempt to do this is to show that the following diagram commutes.

$$\begin{array}{ccccc}
\mathrm{Set}[\mathbb{T} + \mathbb{E}/\phi + \phi] & & & & \mathrm{Set}[\mathbb{T} + \mathbb{E}/\phi + \neg\phi] \\
\parallel & \searrow \pi_{\phi*} & & \nearrow \pi_{\neg\phi}^* & \downarrow \wr \pi_{\mathbb{E}/\phi*} \\
\mathrm{Set}[\mathbb{T} + \phi + \mathbb{E}] & & \mathrm{Set}[\mathbb{T} + \mathbb{E}/\phi] & & \\
\downarrow \pi_{\mathbb{E}*} & \not\parallel & \downarrow \pi_{\mathbb{E}/\phi*} & \nearrow & \downarrow \\
\mathrm{Set}[\mathbb{T} + \phi] & & \mathrm{Set}[\mathbb{T}] & & \mathrm{Set}[\mathbb{T} + \neg\phi] \\
& \searrow \pi_{\phi*} & & \nearrow \pi_{\neg\phi}^* & \\
& & \mathrm{Set}[\mathbb{T}] & &
\end{array}$$

The left half, consisting of direct image functors, does commute. In the right half, where direct and inverse images occur, we can write a ‘‘Beck–Chevalley’’ natural transformation which we would hope to be an isomorphism. For this, however, it cannot suffice to use the information that the corresponding square of geometric morphisms is a pullback with the horizontal geometric morphisms closed embeddings and the right vertical one an equivalence, since this would also be true, for example, if we wrote instead of $\mathrm{Set}[\mathbb{T} + \mathbb{E}/\phi]$

the disjoint union of $\text{Set}[\mathbb{T} + \phi + \mathbb{E}]$ and $\text{Set}[\mathbb{T} + \neg\phi]$ (which is the Artin gluing of the constant functor with value 1).

If we assume the conjecture to be true, we can spell out another special case (containing Proposition 3.8.1) and give a syntactic presentation for the *open mapping cylinder* of a geometric morphism, which is the Artin gluing of its direct image functor.

Proposition 3.8.3. *Let $\pi_{\mathbb{E}} : \text{Set}[\mathbb{T} + \mathbb{E}] \rightarrow \text{Set}[\mathbb{T}]$ be a geometric morphism with chosen syntactic presentation. Assuming Conjecture 3.8.2, the theory $\text{Set}[\mathbb{T} + \langle p \rangle + \mathbb{E}/p]$ is classified by the open mapping cylinder of $\pi_{\mathbb{E}}, \text{Gl}(\pi_{\mathbb{E}*})$.*

Proof. We regard \mathbb{E} as an extension of $\mathbb{T} + \langle p \rangle + p$ and apply the conjecture to the base theory $\mathbb{T} + \langle p \rangle$. What then remains to show is that the composite

$$\text{Set}[\mathbb{T}] \simeq \text{Set}[\mathbb{T} + \langle p \rangle + p] \xrightarrow{\pi_{p*}} \text{Set}[\mathbb{T} + \langle p \rangle] \xrightarrow{\pi_{\neg p}^*} \text{Set}[\mathbb{T} + \langle p \rangle + \neg p] \simeq \text{Set}[\mathbb{T}]$$

is isomorphic to the identity functor. This is true, since $\text{Set}[\mathbb{T} + \langle p \rangle]$ is the product of $\text{Set}[\mathbb{T}]$ with the Sierpiński topos $\text{Set}[\langle p \rangle]$, which can indeed be obtained as the Artin gluing along the identity functor. \square

Remark 3.8.4. There is a dual notion to conditional extensions, where a model involves a model extension in some closed subtopos instead of an open one, and one could make a dual conjecture about replacing a closed subtopos, using the inverse image part of a geometric morphism. Namely, if \mathbb{E} is an extension (without function symbols) of $\mathbb{T} + \neg\phi$ (where ϕ is a closed geometric formula and $\neg\phi$ is the axiom $\phi \vdash_{\square} \perp$), then we would denote $\mathbb{E}/\neg\phi$ the extension of \mathbb{T} consisting of the following.

- For every sort S of \mathbb{E} , a sort S and the axioms $\phi \vdash_{x, x' : S} x = x'$ and $\phi \vdash_{\square} \exists x : S. \top$.
- For every relation symbol $R \subseteq \vec{S}$ of \mathbb{E} , a relation symbol $R \subseteq \vec{S}$ and the axiom $\phi \vdash_{\vec{x} : \vec{S}} R(\vec{x})$.
- For every axiom $\psi \vdash_{\Gamma} \chi$, the axiom $\psi \vdash_{\Gamma} \chi \vee \psi$.

4 Theories of presheaf type

4.1 Introduction

In this section, we develop some results, to be used in the next section, concerning geometric theories whose classifying topos is of the form

$$\mathrm{Set}[\mathbb{T}] \simeq \mathrm{PSh}(C),$$

called theories *of presheaf type*. This class of theories is of great significance for the problem of finding a concise syntactic presentation of a given topos, since it offers the following shortcut. Whenever we know that a theory \mathbb{T} is of presheaf type, there is a canonical (presheaf) site of definition for $\mathrm{Set}[\mathbb{T}]$ defined in a purely categorical way from the category $\mathbb{T}\text{-mod}(\mathrm{Set})$ of Set-based models of \mathbb{T} . Namely, \mathbb{T} is of presheaf type if and only if

$$\mathrm{Set}[\mathbb{T}] \simeq [\mathbb{T}\text{-mod}(\mathrm{Set})_c, \mathrm{Set}],$$

where $\mathbb{T}\text{-mod}(\mathrm{Set})_c$ denotes the category of compact objects in $\mathbb{T}\text{-mod}(\mathrm{Set})$, also known as the finitely presentable models of \mathbb{T} . In particular, a theory of presheaf type is fully determined by its Set-based models, in contrast to the fact that a general geometric theory can have no Set-based models at all without being inconsistent.

To make use of this feature, we need ways to recognize theories of presheaf type without having to construct presheaf sites for their classifying toposes by hand. The basic (and already very useful) result of this kind is that a theory is of presheaf type if its axioms satisfy certain syntactic restrictions, such as, in the simplest case, algebraic theories. If a given theory does not meet this condition, one can of course try to replace the problematic axioms by equivalent axioms of the required form, or, much more drastically, to transform the whole theory into a Morita equivalent theory satisfying the syntactic restrictions, using equivalence extensions to introduce and eliminate function symbols, relation symbols and sorts. However, it is not clear how to find such a transformation if it is not obvious from the theory.

Instead, one can also regard \mathbb{T} as a quotient of a theory \mathbb{T}_0 which contains only syntactically simple axioms. Then, by the duality between quotient theories and subtoposes, the additional axioms of \mathbb{T} correspond to a Grothendieck topology on the canonical presheaf site for $\mathrm{Set}[\mathbb{T}_0]$. If this topology is rigid, meaning that its sheaves are just the presheaves on some smaller presheaf site and therefore \mathbb{T} is again of presheaf type, we speak of a *rigid-topology quotient*. For testing this rigidity condition, it is important to have a convenient description of the topology induced by the additional axioms, which we provide in Theorem 4.3.1.

This strategy can also be used to show that certain classes of axioms, such as axioms stated in the empty context, can be added to *any* theory of presheaf type without destroying the presheaf type property. This allows us, for example, to start with a syntactically simple base theory, apply some equivalence extension to it (possibly involving syntactically complex axioms), and *then* add an axiom that is known to always preserve presheaf type. This is in contrast to the syntactically simple axioms, like algebraic ax-

ioms, which are only harmless in a base theory but not when added to an arbitrary theory of presheaf type, as we will see.

We further explore the possibilities of building up theories of presheaf type incrementally by looking at extensions involving function symbols. While adding finitely many constant symbols does in fact always preserve presheaf type, we quickly obtain negative results after that. Not only can countably many constant symbols or a single unary function symbol destroy presheaf type, they can even do so when added to a theory which is trivial up to Morita equivalence, that is, classified by Set .

The following table summarizes our findings about which syntactic forms of extensions always preserve presheaf type and which can destroy presheaf type.

Extension	Always preserves presheaf type	Reference
Algebraic axiom	no	Example 4.5.4
Axiom in empty context	yes	Corollary 4.4.2
Countably many axioms in empty context	no	Remark 4.4.4
Any number of negated axioms	yes	Corollary 4.4.1
Constant symbol	yes	Proposition 4.4.5
Function symbol	no	Example 4.5.3
Countably many constant symbols	no	Examples 4.5.2 and 4.5.3

4.2 Background

A geometric theory \mathbb{T} is *of presheaf type* if it admits a universal model in some presheaf topos, that is,

$$\text{Set}[\mathbb{T}] \simeq \text{PSh}(C)$$

for some small category C . To discuss this notion properly, we first have to recall the definition of compact objects.

Definition 4.2.1. Let C be a category with all filtered colimits. An object X of C is *compact* if $\text{Hom}(X, -) : C \rightarrow \text{Set}$ preserves filtered colimits. We denote the full subcategory of C on the compact objects by C_c .

The compact objects in $\mathbb{T}\text{-mod}(\text{Set})$ are usually called the *finitely presentable* models of \mathbb{T} , but we will simply call them the compact models and denote the category of these models by

$$\mathbb{T}\text{-mod}(\text{Set})_c.$$

For this to make sense, we must show that the category $\mathbb{T}\text{-mod}(\text{Set})$ has all filtered colimits. We show this for an arbitrary (Grothendieck) topos \mathcal{E} instead, to illustrate how well-behaved models of geometric theories are with respect to filtered colimits.

Lemma 4.2.2. *Let \mathbb{T} be a geometric theory and \mathcal{E} be a (Grothendieck) topos. Then the category $\mathbb{T}\text{-mod}(\mathcal{E})$ admits all filtered colimits, and for every geometric formula-in-context ϕ , the interpretation functor*

$$\mathbb{T}\text{-mod}(\mathcal{E}) \rightarrow \mathcal{E}, \quad M \mapsto \llbracket \phi \rrbracket_M$$

preserves filtered colimits.

Proof. Let $(M_i)_{i \in I}$ be a filtered diagram in $\mathbb{T}\text{-mod}(\mathcal{E})$. We start with the case where \mathbb{T} is the empty theory over some signature Σ , meaning that \mathbb{T} -models are just Σ -structures. For every sort A of Σ , set $\llbracket A \rrbracket_M := \text{colim}_{i \in I} \llbracket A \rrbracket_{M_i}$. For every function symbol $f : A_1 \times \cdots \times A_n \rightarrow B$, we have $\llbracket A_1 \rrbracket_M \times \cdots \times \llbracket A_n \rrbracket_M = \text{colim}_{i \in I} (\llbracket A_1 \rrbracket_{M_i} \times \cdots \times \llbracket A_n \rrbracket_{M_i})$ since filtered colimits commute with finite limits in \mathcal{E} , and the $\llbracket f \rrbracket_{M_i}$ induce an arrow $\llbracket f \rrbracket_M : \llbracket A_1 \rrbracket_M \times \cdots \times \llbracket A_n \rrbracket_M \rightarrow \llbracket B \rrbracket_M$. Similarly, the interpretations $\llbracket R \rrbracket_{M_i} \hookrightarrow \llbracket A_1 \rrbracket_{M_i} \times \cdots \times \llbracket A_n \rrbracket_{M_i}$ of a relation symbol induce an arrow $\llbracket R \rrbracket_M := \text{colim}_{i \in I} \llbracket R \rrbracket_{M_i} \hookrightarrow \llbracket A_1 \rrbracket_M \times \cdots \times \llbracket A_n \rrbracket_M$, which is a monomorphism because filtered colimits commute with pullbacks in \mathcal{E} . In this way, the canonical maps $\llbracket A \rrbracket_{M_i} \rightarrow \llbracket A \rrbracket_M$ obviously constitute Σ -structure homomorphisms. And for a cocone of Σ -structure homomorphisms $M_i \rightarrow \widetilde{M}$, the induced maps $\llbracket A \rrbracket_M \rightarrow \llbracket A \rrbracket_{\widetilde{M}}$ respect the interpretations of function and relation symbols too.

Now let ϕ be a geometric formula (over Σ) in the context $x_1 : A_1, \dots, x_n : A_n$. Then we can prove by induction on the structure of ϕ that $\llbracket \phi \rrbracket_M = \text{colim}_{i \in I} \llbracket \phi \rrbracket_{M_i}$ as subobjects of $\llbracket A_1 \rrbracket_M \times \cdots \times \llbracket A_n \rrbracket_M = \text{colim}_{i \in I} (\llbracket A_1 \rrbracket_{M_i} \times \cdots \times \llbracket A_n \rrbracket_{M_i})$ (using for example that filtered colimits commute with image factorizations). This shows in particular that any axiom $\phi \vdash \psi$ satisfied by all M_i is also satisfied by M , so $\mathbb{T}\text{-mod}(\mathcal{E})$ is closed under filtered colimits in the category of Σ -structures in \mathcal{E} . \square

Let us also prove the following lemma on compact objects, which will be a convenient basic tool for drawing conclusions about compact models in some situations.

Lemma 4.2.3. *Let $F \dashv G$ be adjoint functors $F : C \rightarrow D$, $G : D \rightarrow C$ between categories admitting filtered colimits. If G preserves filtered colimits, then F preserves compact objects.*

Proof. Let $X \in C$ be compact and let Y_i be a filtered diagram in D . Then we have

$$\begin{aligned} \text{Hom}(F(X), \text{colim}_i Y_i) &= \text{Hom}(X, \text{colim}_i G(Y_i)) \\ &= \text{colim}_i \text{Hom}(X, G(Y_i)) = \text{colim}_i \text{Hom}(F(X), Y_i). \end{aligned} \quad \square$$

Now we are ready to understand the usefulness of theories of presheaf type. It consists in the fact that if a geometric theory \mathbb{T} is of presheaf type, then \mathbb{T} is classified by the topos

$$\text{Set}[\mathbb{T}] = \text{PSh}(\mathbb{T}\text{-mod}(\text{Set})_c^{\text{op}}) = [\mathbb{T}\text{-mod}(\text{Set})_c, \text{Set}].$$

That is, if there is any presheaf site for $\text{Set}[\mathbb{T}]$, then the canonically given category $\mathbb{T}\text{-mod}(\text{Set})_c^{\text{op}}$ is also a presheaf site of definition for $\text{Set}[\mathbb{T}]$. We note that the category $\mathbb{T}\text{-mod}(\text{Set})_c$ is essentially small, so that it can actually be used as a site. There is also a simple description of the universal \mathbb{T} -model M in $[\mathbb{T}\text{-mod}(\text{Set})_c, \text{Set}]$, namely, M is the ‘‘tautological’’ model, interpreting a sort A by the functor

$$\mathbb{T}\text{-mod}(\text{Set})_c \rightarrow \text{Set}, \quad M' \mapsto \llbracket A \rrbracket_{M'},$$

and similarly for relation and function symbols [4, Theorem 6.1.1].

For a theory of presheaf type \mathbb{T} , the category $\mathbb{T}\text{-mod}(\text{Set})$ is finitely accessible. This can be a useful criterion for showing that a theory is not of presheaf type.

The basic source of theories of presheaf type is the following. Recall that a geometric theory is a *Horn* theory, if all of its axioms are of the form $\phi \vdash_{\Gamma} \psi$ with ϕ and ψ finite conjunctions of atomic formulas. In particular, algebraic theories are Horn theories. Recall also that in a *cartesian* theory the formulas ϕ and ψ can in addition to finite conjunctions also contain existential quantifiers, but only if these refer to unique existence, provably relative to all preceding axioms in some chosen well-ordering. Then, all Horn theories and more generally all cartesian theories are of presheaf type. For Horn theories, there is even a good general description of the compact models.

Lemma 4.2.4. *Let \mathbb{T} be a Horn theory. Then a model M of \mathbb{T} is compact if and only if it is presented by some Horn formula (finite conjunction of atomic formulas) in context $\vec{x}. \phi$, meaning that there is a natural isomorphism*

$$\text{Hom}(M, -) \cong \llbracket \phi \rrbracket_{-} : \mathbb{T}\text{-mod}(\text{Set}) \rightarrow \text{Set}.$$

Furthermore, every Horn formula in context presents some model.

Proof. See [3, Lemma 4.3]. □

In other words, the compact models of a Horn theory are the ones presented by finitely many generators (with specified sorts) and finitely many relations in the form of atomic formulas.

For understanding the notion of rigid-topology quotients, we should recall the Comparison Lemma, which allows to eliminate some of the objects of a site under certain conditions. If (C, J) is a site and $C' \subseteq C$ is any full subcategory, then we can restrict the Grothendieck topology J to C' , letting a sieve S on an object $c \in C'$ be covering for $J|_{C'}$ if and only if the sieve generated by the arrows of S in the bigger category C is covering for J . In other words, $J|_{C'}$ is the biggest topology on C' such that the inclusion $C' \hookrightarrow C$ preserves covers. A *dense* full subcategory $C' \subseteq C$ of a site (C, J) is one such that every object of C can be covered by (a sieve generated by arrows from) objects in C' .

Theorem 4.2.5 (Comparison Lemma). *If C' is a dense full subcategory of a site (C, J) , then restricting sheaves on C to C' is an equivalence of categories*

$$\text{Sh}(C', J|_{C'}) \simeq \text{Sh}(C).$$

Proof. See [5, Theorem C2.2.3]. □

Now suppose we want to add some axioms to a theory of presheaf type and would like to show that the resulting theory is again of presheaf type. By the duality between quotient theories and subtoposes, and the fact that the latter correspond to Grothendieck topologies on any given site of definition, a quotient extension \mathbb{Q} of a theory of presheaf

type \mathbb{T} induces a topology on $\mathbb{T}\text{-mod}(\text{Set})_c^{\text{op}}$, which we denote $J_{\mathbb{Q}}$. If there is a dense full subcategory of $\mathbb{T}\text{-mod}(\text{Set})_c^{\text{op}}$ with respect to $J_{\mathbb{Q}}$, such that $J_{\mathbb{Q}}$ becomes the trivial topology when restricted to this subcategory, then by the Comparison Lemma, $\mathbb{T} + \mathbb{Q}$ is again of presheaf type. This condition can equivalently be expressed by saying that the *irreducible* objects of $\mathbb{T}\text{-mod}(\text{Set})_c^{\text{op}}$, which admit no covering sieve except the maximal sieve, are dense. A topology with this property is called a *rigid* topology.

Definition 4.2.6. Let \mathbb{T} be a theory of presheaf type. We say that a quotient extension \mathbb{Q} of \mathbb{T} is a *rigid-topology quotient* if the Grothendieck topology $J_{\mathbb{Q}}$ on $\mathbb{T}\text{-mod}(\text{Set})_c^{\text{op}}$ is rigid.

Note that for a quotient extension \mathbb{Q} of an arbitrary geometric theory \mathbb{T} , we would have to fix a site of definition for $\text{Set}[\mathbb{T}]$ before we can ask if the induced topology is rigid.

After proving that some \mathbb{Q} is a rigid-topology quotient, we would probably like to have a description of the compact models of $\mathbb{T} + \mathbb{Q}$. From Lemma 4.2.2 it follows that the full subcategory

$$(\mathbb{T} + \mathbb{Q})\text{-mod}(\text{Set}) \subseteq \mathbb{T}\text{-mod}(\text{Set})$$

is closed under filtered colimits. With this, one easily sees that a $(\mathbb{T} + \mathbb{Q})$ -model which is compact as a \mathbb{T} -model is also compact as a $(\mathbb{T} + \mathbb{Q})$ -model (even without any presheaf type conditions). The converse is part of the following result, which simultaneously answers the question whether \mathbb{Q} being a rigid-topology quotient is necessary for $\mathbb{T} + \mathbb{Q}$ to be again of presheaf type.

Lemma 4.2.7. *A quotient extension \mathbb{Q} of a theory of presheaf type \mathbb{T} is a rigid-topology quotient if and only if $\mathbb{T} + \mathbb{Q}$ is again of presheaf type and additionally every compact $(\mathbb{T} + \mathbb{Q})$ -model is also compact as a \mathbb{T} -model.*

Proof. See [4, Theorem 8.2.6]. □

So for example, take as \mathbb{T} the theory consisting of a sort A and countably many constant symbols $c_n : A$, and let \mathbb{Q} be the axiom

$$\top \vdash_{x,y:A} x = y.$$

Then both \mathbb{T} and $\mathbb{T} + \mathbb{Q}$ are algebraic theories and thus of presheaf type. But in the unique Set -based model of $\mathbb{T} + \mathbb{Q}$, the constants c_n are all identified, which can not happen in a compact model of \mathbb{T} , by Lemma 4.2.4. So \mathbb{Q} is not a rigid-topology quotient.

4.3 Rigid-topology quotients

Let $M \in \mathbb{T}\text{-mod}(\text{Set})_c$ be a compact model of a theory of presheaf type and let

$$\phi \vdash_{\vec{x}: \vec{A}} \psi$$

be an axiom that we would like to add to \mathbb{T} . This axiom is satisfied in the model M if and only if we have $\llbracket \phi \rrbracket_M \subseteq \llbracket \psi \rrbracket_M$. But in any case, given some $\vec{x} \in \llbracket \phi \rrbracket_M$, we can consider the

collection of all arrows $f : M \rightarrow M'$ in $\mathbb{T}\text{-mod}(\text{Set})_c$ that map \vec{x} into $\llbracket \psi \rrbracket_{M'}$, which turns out to be a cosieve on M , since any further model homomorphism $M' \rightarrow M''$ preserves truth of the geometric formula ψ . We now show that these sieves in $\mathbb{T}\text{-mod}(\text{Set})_c^{\text{op}}$ generate the Grothendieck topology corresponding to the axiom $\phi \vdash_{\vec{x}:\vec{A}} \psi$.

Theorem 4.3.1. *Let \mathbb{T} be a theory of presheaf type and let $\mathbb{Q} = \{ \phi \vdash_{\vec{x}:\vec{A}} \psi \}$ be a quotient extension of \mathbb{T} adding a single axiom. Then the Grothendieck topology $J_{\mathbb{Q}}$ on $\mathbb{T}\text{-mod}(\text{Set})_c^{\text{op}}$ is generated by the sieves (cosieves in $\mathbb{T}\text{-mod}(\text{Set})_c$)*

$$S_{\vec{x}} = S_{M, \vec{x}, \psi} := \{ f : M \rightarrow M' \mid f(\vec{x}) \in \llbracket \psi \rrbracket_{M'} \},$$

where $M \in \mathbb{T}\text{-mod}(\text{Set})_c$ and $\vec{x} = (x_1, \dots, x_n) \in \llbracket \phi \rrbracket_M \subseteq \llbracket A_1 \rrbracket_M \times \dots \times \llbracket A_n \rrbracket_M$.

Proof. The axiom $\phi \vdash_{\vec{x}:\vec{A}} \psi$ is satisfied in a model $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ (in any topos) if and only if the inclusion $\llbracket \psi \wedge \phi \rrbracket_M \hookrightarrow \llbracket \phi \rrbracket_M$ of subobjects of $\llbracket A_1 \rrbracket_M \times \dots \times \llbracket A_n \rrbracket_M$ is an isomorphism. For the universal \mathbb{T} -model in $[\mathbb{T}\text{-mod}(\text{Set})_c, \text{Set}]$, this is the inclusion

$$\llbracket \psi \wedge \phi \rrbracket_- \hookrightarrow \llbracket \phi \rrbracket_-$$

of functors $\mathbb{T}\text{-mod}(\text{Set})_c \rightarrow \text{Set}$. The classifying topos of $\mathbb{T} + \mathbb{Q}$ is therefore the greatest subtopos (corresponding to the Grothendieck topology with the fewest covering sieves) with the property that this inclusion is a *local isomorphism*, that is, it becomes an isomorphism when sheafified.

Now, let $M \in \mathbb{T}\text{-mod}(\text{Set})_c$ and let $\vec{x} \in \llbracket \phi \rrbracket_M$ be given, corresponding to an arrow $\text{Hom}(M, -) \rightarrow \llbracket \phi \rrbracket_-$. Then the cosieve $S_{\vec{x}}$ on M , regarded as a subobject of $\text{Hom}(M, -)$ in $[\mathbb{T}\text{-mod}(\text{Set})_c, \text{Set}]$, is the pullback

$$\begin{array}{ccc} \llbracket \psi \wedge \phi \rrbracket_- & \hookrightarrow & \llbracket \phi \rrbracket_- \\ \uparrow & \lrcorner & \uparrow \\ S_{\vec{x}} & \hookrightarrow & \text{Hom}(M, -). \end{array}$$

Thus, if the top row is a local isomorphism, then the bottom row is a local isomorphism for all M and \vec{x} . But the converse is also true, since any local section $\tilde{x} \in a(\llbracket \phi \rrbracket_-)(M)$ of the sheafification of $\llbracket \phi \rrbracket_-$ is locally given by some $\vec{x} \in \llbracket \phi \rrbracket_{M'}$, which then locally lies in $\llbracket \psi \wedge \phi \rrbracket_-$. From this, the statement follows, because the covering sieves of a Grothendieck topology are precisely those subobjects of representable presheaves which are local isomorphisms. \square

Remark 4.3.2. Theorem 4.3.1 is closely related to [4, Theorem 8.1.10], which describes the topology $J_{\mathbb{Q}}$ by only one generating sieve per axiom, assuming that the axioms are given in a certain form, basically consisting of formulas that present compact models. While this yields much more concise descriptions of the topology, it seems harder to apply to concrete theories, as it might be nontrivial to determine whether a formula presents a model (this property is characterized in [4, Theorem 6.1.13] as being an irreducible object of the syntactic site). Also, for our main purpose of recognizing the topology $J_{\mathbb{Q}}$ as rigid, the big number of covering sieves $S_{\vec{x}}$ in Theorem 4.3.1 will actually be quite convenient and can therefore be seen as a feature in this context.

If the quotient extension \mathbb{Q} contains more than one axiom, the topology $J_{\mathbb{Q}}$ is of course generated by the union of all the sets of generators for the individual axioms. (So there is one generating cosieve for every axiom, compact model and appropriate family of elements of the model.) It should be noted that the collection of cosieves of the form $S_{\vec{x}}$ satisfies the pullback-stability axiom of a Grothendieck topology (in the terminology of [5, C2.1], it is a (sifted) coverage). Indeed, if $g : M \rightarrow M'$ is a homomorphism between compact models then the push-forward of a cosieve $S_{\vec{x}}$ on M for a family of elements $\vec{x} \in \llbracket \phi \rrbracket_M$ is simply the cosieve $S_{g(\vec{x})}$ on M' , and we do have $g(\vec{x}) \in \llbracket \phi \rrbracket_{M'}$.

As a first application of Theorem 4.3.1, we can give a short proof of the following result.

Corollary 4.3.3 (see [4, Theorem 8.2.5]). *Let \mathbb{T} be a theory of presheaf type and \mathbb{Q} a quotient extension of \mathbb{T} . Then a compact model of \mathbb{T} is $J_{\mathbb{Q}}$ -irreducible as an object of $\mathbb{T}\text{-mod}(\text{Set})_c^{\text{op}}$ if and only if it satisfies the axioms of \mathbb{Q} .*

Proof. A compact model M of \mathbb{T} satisfies an axiom $\phi \vdash_{\vec{x}:\vec{A}} \psi$ if and only if for every family of elements $\vec{x} \in \llbracket \phi \rrbracket_M$, the cosieve $S_{\vec{x}}$ from Theorem 4.3.1 is the maximal cosieve on M . This shows the “only if” direction. For the “if” direction, one can check that saturating a collection of sieves with respect to the transitivity condition of Grothendieck topologies (preserves pullback-stability of the collection and) can’t produce any nontrivial covers of an object if there were none before. \square

Remark 4.3.4. We can understand the cosieves $S_{\vec{x}}$ from Theorem 4.3.1 as *operations* correcting the failure of a single instance of one of the axioms of \mathbb{Q} . Showing that $J_{\mathbb{Q}}$ is rigid then amounts to providing an algorithm that turns any compact \mathbb{T} -model into a $(\mathbb{T} + \mathbb{Q})$ -model using these operations, as follows. Starting with a compact \mathbb{T} -model M which is not a model of $\mathbb{T} + \mathbb{Q}$, we have to pick an axiom $\phi \vdash_{\vec{x}:\vec{A}} \psi$ of \mathbb{Q} and a family \vec{x} of elements of (the appropriate sorts of) M with $\phi(\vec{x})$ but not $\psi(\vec{x})$. Then, if we are lucky, the cosieve $S_{\vec{x}}$ is generated by a single arrow

$$M \rightarrow M',$$

meaning that we have reduced the problem of covering M by irreducibles to covering M' by irreducibles by the transitivity property of the Grothendieck topology $J_{\mathbb{Q}}$. If we can repeat this procedure a finite number of times and arrive at a model that satisfies all axioms in \mathbb{Q} , then we are done; we have covered M by a single irreducible object.

It can of course happen that $S_{\vec{x}}$ is not a principal cosieve, so that we need multiple arrows

$$M \rightarrow M'_i$$

to generate it. Then $S_{\vec{x}}$ should perhaps be seen as a nondeterministic operation, that will turn M into one of the M'_i , but we don’t know in advance which one it will be. In this case, we have to deal with all M'_i , and our algorithm must terminate in the sense that any possible path of execution is finite. (On the other hand, we can also be even *more* lucky, in that $S_{\vec{x}}$ is the empty cosieve on M . Then we are immediately done with the current branch.)

Here is a first illustration of this strategy.

Example 4.3.5. Start with the theory \mathbb{T} with two sorts and a function symbol $A \rightarrow B$, and add to it the axiom

$$\mathbb{T} \vdash_{y:B} \exists x:A. f(x) = y,$$

requiring f to be surjective. We already know that the resulting theory is of presheaf type, since it is Morita equivalent to the theory with just one sort A and an equivalence relation $\sim \subseteq A \times A$ by Example 2.4.2, and this is a Horn theory. But let us show that the surjectivity axiom is a rigid-topology quotient of \mathbb{T} .

By Lemma 4.2.4, a model $M = (A \rightarrow B) \in \mathbb{T}\text{-mod}(\text{Set})$ is compact if and only if both A and B are finite sets. Given such a compact model $(A \rightarrow B)$ and an element $y \in B$, the cosieve S_y on $(A \rightarrow B)$ from Theorem 4.3.1 is generated by the single arrow

$$\begin{array}{c} (A \rightarrow B) \\ \downarrow \\ (A \sqcup \{x\} \rightarrow B), \end{array}$$

where x is sent to y in the new model (which is clearly still compact), since every model homomorphism

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g_A & & \downarrow g_B \\ A' & \xrightarrow{f'} & B' \end{array}$$

with $\exists x \in A'. f'(x) = g_B(y)$ factors through $(A \sqcup \{x\} \rightarrow B)$. The model $(A \rightarrow B)$ can thus be covered (in the topology induced by the surjectivity axiom) by the model $(A \sqcup \{x\} \rightarrow B)$. We can go on by creating preimages of other elements of B , until we have covered $(A \rightarrow B)$ by a model satisfying the new axiom (since B is finite).

Lemma 4.3.6. *If \mathbb{T} is of presheaf type, \mathbb{Q}_1 is a rigid-topology quotient of \mathbb{T} and \mathbb{Q}_2 is a rigid-topology quotient of $\mathbb{T} + \mathbb{Q}_1$, then $\mathbb{Q}_1 + \mathbb{Q}_2$ is a rigid-topology quotient of \mathbb{T} .*

Proof. This follows easily from Lemma 4.2.7: $\mathbb{T} + \mathbb{Q}_1 + \mathbb{Q}_2$ is of presheaf type and every compact $(\mathbb{T} + \mathbb{Q}_1 + \mathbb{Q}_2)$ -model is compact as a $(\mathbb{T} + \mathbb{Q}_1)$ -model and hence as a \mathbb{T} -model.

Note that arguing more directly by the transitivity of $J_{\mathbb{Q}_1 + \mathbb{Q}_2}$ is not so simple, as we would still have to relate the topology induced by \mathbb{Q}_2 on $(\mathbb{T} + \mathbb{Q}_1)\text{-mod}(\text{Set})_c$ with that on $\mathbb{T}\text{-mod}(\text{Set})_c$. \square

While we can assume, after choosing a representing set of axioms for the quotient extension \mathbb{Q}_2 , that its base theory is \mathbb{T} instead of $\mathbb{T} + \mathbb{Q}_1$, the assumptions of Lemma 4.3.6 are not at all the same as requiring \mathbb{Q}_1 and \mathbb{Q}_2 to be rigid-topology quotients of \mathbb{T} independently. In fact, $\mathbb{Q}_1 + \mathbb{Q}_2$ can fail to be a rigid-topology quotient of \mathbb{T} in this case, as we will see in Example 4.5.5. By the same method as there, one can easily construct quotient extensions $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ such that the sum of any proper subset induces a rigid topology, but adding all n even destroys presheaf type.

4.4 Syntactic constructions preserving presheaf type

We now investigate in which ways a theory of presheaf type can be extended such that the resulting theory is again of presheaf type. At first sight, one might hope for a result like “an algebraic (or Horn, or even cartesian) extension preserves presheaf type”, but unfortunately, this is far too optimistic. Indeed, if we have any geometric axiom $\phi \vdash_{\vec{x}:\vec{A}} \psi$ we would like to add to a theory, we can instead first perform an extension by definitions (which surely preserves presheaf type), defining $R_\phi, R_\psi \subseteq \vec{A}$ by ϕ and ψ respectively, and then add the Horn axiom $R_\phi \vdash_{\vec{x}:\vec{A}} R_\psi$. Example 4.5.4 shows that also an algebraic axiom can destroy presheaf type.

Corollary 4.4.1 (see [4, Theorem 8.2.8]). *Adding arbitrarily many axioms of the form*

$$\phi \vdash_{\vec{x}:\vec{A}} \perp$$

to a theory of presheaf type is always a rigid-topology quotient.

Proof. Let \mathbb{T} be a theory of presheaf type, $\mathbb{Q} = \{\phi^i \vdash_{\vec{x}^i:\vec{A}^i} \perp \mid i \in I\}$ and $M \in \mathbb{T}\text{-mod}(\text{Set})_c$. If there is some $i \in I$ and $\vec{x} \in \llbracket \phi^i \rrbracket_M$, then

$$S_{\vec{x}} = \{f : M \rightarrow M' \mid f(\vec{x}) \in \llbracket \perp \rrbracket_{M'} = \emptyset\}$$

is the empty cosieve on M and we are done. But otherwise M was already a model of $\mathbb{T} + \mathbb{Q}$, i.e. irreducible itself. \square

Corollary 4.4.2. *Adding finitely many axioms $\phi \vdash_{\square} \psi$ with empty context to a theory of presheaf type is always a rigid-topology quotient.*

Proof. For one axiom $\phi \vdash_{\square} \psi$ with empty context, Theorem 4.3.1 says that any compact model M that satisfies the closed formula ϕ is covered by all arrows (necessarily preserving truth of ϕ) to compact models that also satisfy ψ , as required. And any model that does not satisfy ϕ satisfies the new axiom anyway. For more than one axiom of this form, we can use Lemma 4.3.6. \square

Corollary 4.4.3. *Any propositional theory with finitely many axioms is of presheaf type.*

Proof. This immediately follows from Corollary 4.4.2, as a propositional theory has no other contexts than the empty one. \square

Remark 4.4.4. Corollary 4.4.2 becomes false if we omit the word “finitely”. Indeed, it would otherwise imply that all propositional theories are of presheaf type (see Example 4.5.3 for one which is not). In particular, Lemma 4.3.6 becomes wrong if instead of two consecutive rigid-topology quotients we consider the sum $\mathbb{T} + \sum_{i \in \mathbb{N}} \mathbb{Q}_i$ of an infinite sequence of quotient extensions where every \mathbb{Q}_n is a rigid-topology quotient of $\mathbb{T} + \sum_{i=0}^{n-1} \mathbb{Q}_i$.

Proposition 4.4.5. *Let \mathbb{T} be a theory of presheaf type and A a sort of \mathbb{T} . Then the theory $\mathbb{T} + (c : A)$ obtained from \mathbb{T} by introducing a new constant symbol $c : A$ is also of presheaf type. Moreover, a model of $\mathbb{T} + (c : A)$ in Set is compact if and only if the underlying \mathbb{T} -model is compact.*

Proof. Let $\text{Set}[\mathbb{T}]$ be a classifying topos for \mathbb{T} with universal model $M_{\mathbb{T}}$. Then $\mathbb{T} + (c : A)$ is classified by the slice topos $\text{Set}[\mathbb{T}]/\llbracket A \rrbracket_{M_{\mathbb{T}}}$, see [6, Exercise X.3]. But if $\text{Set}[\mathbb{T}] = \text{PSh}(C)$ is a presheaf topos, then we can use the formula [6, Exercise III.8]

$$\text{PSh}(C)/X \simeq \text{PSh}\left(\int_C X\right)$$

for $X = \llbracket A \rrbracket_{M_{\mathbb{T}}}$ to see that $\mathbb{T} + (c : A)$ is of presheaf type.

For the compact models, we observe that $(\mathbb{T} + (c : A))\text{-mod}(\text{Set})$ is equivalent, as a category over $\mathbb{T}\text{-mod}(\text{Set})$, to the category $\int^{\mathbb{T}\text{-mod}(\text{Set})} \llbracket A \rrbracket_-$ of elements of the interpretation functor $\llbracket A \rrbracket_- : \mathbb{T}\text{-mod}(\text{Set}) \rightarrow \text{Set}$. This functor preserves filtered colimits by Lemma 4.2.2, so Lemma 4.4.6 below tells us exactly what we need. \square

Lemma 4.4.6. *Let C be a category with filtered colimits and let $F : C \rightarrow \text{Set}$ be a functor preserving filtered colimits. Then the category of elements of F , $\int^C F$, has filtered colimits too, and the projection functor $U : \int^C F \rightarrow C$ preserves and reflects compact objects.*

Proof. Let $(c_i, x_i)_{i \in I}$, where $x_i \in F(c_i)$, be a filtered diagram in $\int^C F$. It is clear that the x_i all represent the same element in $\text{colim}_i F(c_i) \cong F(\text{colim}_i c_i)$, let $x^* \in F(\text{colim}_i c_i)$ be this element. Then we have a cocone on $(c_i, x_i)_{i \in I}$ with apex $(\text{colim}_i c_i, x^*)$, and it is easy to check that this is the colimit in $\int^C F$. This also shows that the functor $U : \int^C F \rightarrow C$ preserves and reflects filtered colimits.

Let (c, x) be a compact object of $\int^C F$, we want to show that c is compact in C . So let $(c_i)_{i \in I}$ be a filtered diagram in C and fix a morphism

$$f : c \rightarrow \text{colim}_{i \in I} c_i,$$

then we have to show that there is exactly one $[g] \in \text{colim}_i \text{Hom}(c, c_i)$ that induces f . If we want to use the compactness of (c, x) , we need to construct a filtered diagram in $\int^C F$. Consider the element $f_*x \in F(\text{colim}_i c_i) \cong \text{colim}_i F(c_i)$ and pick a representative $x_{i_0} \in F(c_{i_0})$ for it. Since I is filtered, the comma category $(i_0 \downarrow I)$ is also filtered and the functor $(i_0 \downarrow I) \rightarrow I$ is final. This means that we have $\text{colim}_i c_i = \text{colim}_{i_0 \rightarrow i} c_i$ and $\text{colim}_i \text{Hom}(c, c_i) = \text{colim}_{i_0 \rightarrow i} \text{Hom}(c, c_i)$, so we can replace the diagram $(c_i)_{i \in I}$ with $(c_i)_{(i_0 \rightarrow i) \in (i_0 \downarrow I)}$ and assume without loss of generality that i_0 is initial in I . Then setting $x_i := (! : i_0 \rightarrow i)_*x_{i_0} \in F(c_i)$ yields a filtered diagram $(c_i, x_i)_{i \in I}$ in $\int^C F$. We also have

$$\tilde{f} : (c, x) \rightarrow \text{colim}_{i \in I} (c_i, x_i)$$

with $U(\tilde{f}) = f$ by our original choice of x_{i_0} . So by assumption, there is a $\tilde{g} : (c, x) \rightarrow (c_i, x_i)$ for some i that induces \tilde{f} . Then $g := U(\tilde{g})$ induces f . On the other hand, if $g_1 : c \rightarrow c_{i_1}$ and $g_2 : c \rightarrow c_{i_2}$ both induce f , then we have $[g_{1*}x] = [g_{2*}x] = [x_{i_0}] \in \text{colim}_i F(c_i)$, in particular $[g_{1*}x] = [x_{i_1}]$ and $[g_{2*}x] = [x_{i_2}]$, so we can prolong g_1 to some $g'_1 : c \rightarrow c'_{i_1}$ such that $g'_{1*}x = x_{i_1}$, and similarly for g_2 , meaning that we have

$\tilde{g}_1 : (c, x) \rightarrow (c_{i'_1})$ and $\tilde{g}_2 : (c, x) \rightarrow (c_{i'_2})$ that both induce \tilde{f} . Then we can conclude $[\tilde{g}_1] = [\tilde{g}_2] \in \text{colim}_i \text{Hom}((c, x), (c_i, x_i))$, and from this it follows that $[g_1] = [g'_1] = [g_2] = [g'_2] \in \text{colim}_i \text{Hom}(c, c_i)$.

Now let $c \in C$ be compact and let $x \in F(c)$, we want to show that (c, x) is compact in $\int^C F$. So let $(c_i, x_i)_{i \in I}$ be a filtered diagram in $\int^C F$ and let

$$\tilde{f} : (c, x) \rightarrow \text{colim}_{i \in I} (c_i, x_i) = (\text{colim}_{i \in I} c_i, x^*)$$

be given. That is, we have $f : c \rightarrow \text{colim}_i c_i$ with $f_*x = x^*$. This f is induced by some $g : c \rightarrow c_i$, implying that $[g_*x] = x^* \in \text{colim}_i F(c_i)$, so it is also induced by some $g' : c \rightarrow c_{i'}$ with $g'_*x = x_{i'}$. This means that we have $\tilde{g} : (c, x) \rightarrow (c_{i'}, x_{i'})$ inducing \tilde{f} . And for \tilde{g}_1, \tilde{g}_2 both inducing \tilde{f} , we immediately see that $[U(\tilde{g}_1)] = [U(\tilde{g}_2)] \in \text{colim}_i \text{Hom}(c, c_i)$ and therefore $[\tilde{g}_1] = [\tilde{g}_2] \in \text{colim}_i \text{Hom}((c, x), (c_i, x_i))$. \square

Example 4.4.7. From Example 2.4.4 we know that the theory \underline{A} , where we have just imported a set A , is Morita equivalent to the empty theory (over the empty signature). In particular, it is of presheaf type and we can apply Proposition 4.4.5 to conclude that the theory $c : \underline{A}$, where we have added one more constant symbol is also of presheaf type. It is classified by Set/A .

It is also Morita equivalent to a disjoint disjunction of an A -indexed family of proposition symbols,

$$\top \vdash_{\square} \bigvee_{a \in A} p_a \quad \text{and} \quad p_a \wedge p_{a'} \vdash_{\square} \perp \quad (\text{for } a \neq a' \in A),$$

by identifying p_a with $c = c_a$. This way, we can easily see that the theory is of presheaf type using Corollary 4.4.2 for the first axiom and Corollary 4.4.1 for the A -indexed family of negated axioms.

Example 4.4.8. Consider the theory of a surjective function $f : A \twoheadrightarrow B$, which is of presheaf type by Example 2.4.2 or Example 4.3.5. By Proposition 4.4.5, it is still of presheaf type after adding a constant symbol $c : B$.

When we translate this extension to the theory of an equivalence relation $\sim \subseteq A \times A$, we obtain a relation symbol $R \subseteq A$ with the axioms

$$R(x) \wedge x \sim x' \vdash_{x, x' : A} R(x'), \quad R(x) \wedge R(x') \vdash_{x, x' : A} x \sim x', \quad \top \vdash_{\square} \exists x : A. R(x).$$

The existential quantifier in the last axiom is not cartesian, but the context of the axiom is empty (and all other axioms of the theory are Horn), so we again find that the theory is of presheaf type using Corollary 4.4.2. However, if the sort A and the equivalence relation \sim are instead given as part of a theory of presheaf type which is not cartesian, then the Horn axioms on R can not be argued away so easily, but we can still introduce the quotient $B = A/\sim$ and add a constant symbol $c : B$ without destroying presheaf type.

4.5 Some counterexamples

We now give a number of examples of non-presheaf-type theories, to show that the above positive results are optimal in certain aspects and to illustrate the various techniques useful for recognizing such theories. Our first example shows that even a very simple, finite theory can fail to be of presheaf type.

Example 4.5.1. Consider the theory

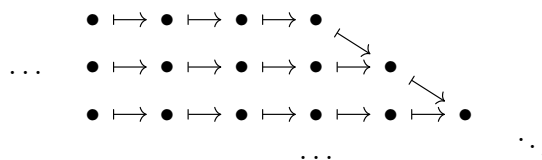
$$\mathbb{T} = \left\{ A \xrightarrow{\varphi} f \right\}$$

consisting of one sort A , a unary function symbol $f : A \rightarrow A$ and one axiom $\top \vdash_{y:A} \exists x:A. f(x) = y$ stating that f is surjective. We show that $\mathbb{T}\text{-mod}(\text{Set})$ is not finitely accessible, implying that \mathbb{T} is not of presheaf type.

Specifically, we show that the model $(\mathbb{Z}, +1)$, where $A = \mathbb{Z}$ and $f(n) = n + 1$, is not a filtered colimit of compact models. Consider the models

$$M_n = \left(\{ (x, y) \in \mathbb{Z}^2 \mid y \leq \min(n, -x) \}, (x, y) \mapsto (x + 1, \min(y, -x - 1)) \right),$$

which can be pictured like this.



The inclusions $M_n \hookrightarrow M_{n+1}$ (adding another row on top) are model homomorphisms, and we have a homomorphism

$$(\mathbb{Z}, +1) \rightarrow M_\omega := \text{colim}_n M_n, \quad k \mapsto (k, -k),$$

that does not factor through any M_n . This shows that $(\mathbb{Z}, +1)$ is not compact. But any map $M \rightarrow (\mathbb{Z}, +1)$ from a nonempty model M is necessarily surjective, so the composite $M \rightarrow (\mathbb{Z}, +1) \rightarrow M_\omega$ does not factor through any M_n either, showing that $(\mathbb{Z}, +1)$ is not a colimit of compact models.

Another method to show that a theory has “too few compact models” to be of presheaf type is illustrated by the next example. In it, we see that adding countably many constant symbols of the same sort can destroy presheaf type (in contrast to Proposition 4.4.5). An even more drastic version of this statement will be one of the conclusions of Example 4.5.3.

Example 4.5.2. Start with the theory of a surjective function $f : A \rightarrow B$ as in Example 4.4.8, but then add countably many constant symbols $c_0, c_1, \dots : B$ instead of just a single one. We show that this theory \mathbb{T} is not of presheaf type by investigating its compact models.

$$A \rightarrow B \ni c_0, c_1, \dots$$

Let \mathbb{T}' be the theory of inhabited sets, with one sort A and the axiom $\top \vdash_{\square} (\exists x : A. \top)$. Then we can regard \mathbb{T} as an extension of \mathbb{T}' and the forgetful functor

$$U : \mathbb{T}\text{-mod}(\text{Set}) \rightarrow \mathbb{T}'\text{-mod}(\text{Set}), \quad (A \rightarrow B) \mapsto A$$

has a right adjoint $A' \mapsto (A' \rightarrow \{*\})$, which preserves filtered colimits by their description in Lemma 4.2.2. Thus, U preserves compact objects by Lemma 4.2.3. But the compact objects of $\mathbb{T}'\text{-mod}(\text{Set})$ are the finite nonempty sets, so, in summary, for any compact model $(A \rightarrow B)$ of \mathbb{T} , A is a finite set. This means that B is finite too, from which we can in particular conclude that the geometric formula

$$\phi := \bigvee_{i \neq j} (c_i = c_j)$$

is satisfied in all compact models. If \mathbb{T} was of presheaf type, this would mean that ϕ is satisfied in the universal model and therefore provable in \mathbb{T} . This is not the case, as witnessed by the model $(\mathbb{N} \rightarrow \mathbb{N})$ with $c_n = n$.

The following example will allow us to draw several conclusions. The technique used here is to compare a theory to another theory with the same Set -based models which is known to be of presheaf type.

Example 4.5.3. Given two sets A and B , there are (at least) two different theories “of a map from A to B ”. One is

$$\langle f : \underline{B^A} \rangle,$$

where we import (see Example 2.4.4) the set B^A of functions from A to B and add a constant symbol f of this sort; it is of presheaf type as in Example 4.4.7. The other is

$$\langle f : \underline{A} \rightarrow \underline{B} \rangle,$$

where we import A and B separately and add a unary function symbol as indicated. For both theories, the category of models in Set is (up to equivalence) the discrete category with B^A objects. So if $\langle f : \underline{A} \rightarrow \underline{B} \rangle$ was of presheaf type too, then this would imply that the two theories are Morita-equivalent.

Specialize to the case $A = \mathbb{N}$ and $B = 2 = \{0, 1\}$. Then $\langle f : \underline{2^{\mathbb{N}}} \rangle$ is classified by $\text{Set}/2^{\mathbb{N}}$, or equivalently, by $\text{Sh}(X)$, where X is the discrete space with $2^{\mathbb{N}}$ points. But by [6, Exercise VIII.10], $\langle f : \underline{\mathbb{N}} \rightarrow \underline{2} \rangle$ is classified by $\text{Sh}(2^{\mathbb{N}})$, where $2^{\mathbb{N}}$ is the Cantor space. Since these two sober topological spaces are not homeomorphic, $\langle f : \underline{\mathbb{N}} \rightarrow \underline{2} \rangle$ is not of presheaf type.

Another way to see that $\langle f : \underline{2^{\mathbb{N}}} \rangle$ and $\langle f : \underline{\mathbb{N}} \rightarrow \underline{2} \rangle$ are not Morita equivalent is as follows. First, as in Example 4.4.7, a constant symbol in an imported set, such as $f : \underline{2^{\mathbb{N}}}$, can be replaced by a disjoint disjunction of $2^{\mathbb{N}}$ proposition symbols. On the other hand, the function symbol $f : \underline{\mathbb{N}} \rightarrow \underline{2}$ can be replaced by countably many constant symbols $c_n : \underline{2}$ and therefore by countably many decidable propositions (disjoint binary disjunctions). In this way, one sees that the models of both theories in any topos \mathcal{E} form discrete categories, namely, $\langle f : \underline{2^{\mathbb{N}}} \rangle$ classifies decompositions of \mathcal{E} into $2^{\mathbb{N}}$ clopen

subtoposes, while $\langle f : \mathbb{N} \rightarrow \underline{2} \rangle$ classifies countable families of compositions into two clopen subtoposes. Now, we can consider the topological spaces $X_n := \{0, \dots, n\}$ (with the discrete topology) and $X_\omega := \mathbb{N} \cup \{\infty\}$, the one-point compactification of \mathbb{N} , and the continuous maps

$$X_\omega \rightarrow X_n, \quad x \mapsto \min(x, n).$$

Then we see that every model of $\langle f : \underline{2}^{\mathbb{N}} \rangle$ in X_ω is (isomorphic to) the pullback of a model in some X_n , because any open set of X_ω containing ∞ is cofinite, but this is not the case for $\langle f : \mathbb{N} \rightarrow \underline{2} \rangle$, because we can choose a family of partitions like $X_\omega = \{0, \dots, n\} \sqcup \{n+1, \dots, \infty\}$.

In summary, we can record the following.

- (i) Adding a single unary function symbol (such as $f : \mathbb{N} \rightarrow \underline{2}$) to a theory of presheaf type (even a theory that is classified by \mathbf{Set}) can destroy presheaf type.
- (ii) Adding countably many constant symbols (such as $c_n : \underline{2}$) to a theory of presheaf type (even a theory that is classified by \mathbf{Set}) can destroy presheaf type.
- (iii) A propositional theory (such as the theory of a countable family of decidable propositions, with $p_n \vee q_n$ and $p_n \wedge q_n \vdash_{\square} \perp$) can fail to be of presheaf type. (There are of course many other examples of this.)
- (iv) Adding countably many closed geometric formulas (such as $p_n \vee q_n$ in the previous item, where all other axioms were negated ones, which can't destroy presheaf type) can destroy presheaf type.

Example 4.5.4. Let us show that adding an algebraic axiom can destroy presheaf type. We do this by showing that adding an algebraic axiom is, up to Morita equivalence, at least as expressive as adding an arbitrary family ϕ_i of closed geometric formulas as axioms. We can then for example use $\phi_n = (p_n \vee q_n)$ to obtain the theory of countably many decidable propositions from a theory with only negated axioms, as in example 4.5.3.

So let \mathbb{T} be a theory with closed geometric formulas $\phi_i, i \in I$. Import the set $I \sqcup \{*\}$. We can define (by a geometric formula) an equivalence relation \sim on $I \sqcup \{*\}$ such that $c_i \sim c_* \dashv\vdash_{\square} \phi_i$. Introduce the quotient $I \sqcup \{*\} \twoheadrightarrow A$ by \sim . These equivalence extensions allow us to formulate the algebraic axiom

$$\mathbb{T} \vdash_{x,y:A} x = y,$$

which is syntactically equivalent (as a quotient extension) to $\mathbb{T} \vdash_{x:I \sqcup \{*\}} x \sim c_*$, and therefore to the set of axioms $\{\mathbb{T} \vdash_{\square} \phi_i \mid i \in I\}$.

Example 4.5.5. Here we give an example of two rigid-topology quotients $\mathbb{Q}_1, \mathbb{Q}_2$ of the same theory of presheaf type \mathbb{T} such that $\mathbb{Q}_1 + \mathbb{Q}_2$ is not a rigid-topology quotient again and $\mathbb{T} + \mathbb{Q}_1 + \mathbb{Q}_2$ is not even of presheaf type. Start with the theory \mathbb{T} of a sequence of maps, infinite to the left:

$$\dots \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1 \xrightarrow{f_0} A_0.$$

The compact Set-models of \mathbb{T} are those with finitely many elements in total. Consider the quotient extension \mathbb{Q}_1 requiring every second map to be surjective:

$$\mathbb{T} \vdash_{y:A_n} \exists x:A_{n+1}. f_n(x) = y \quad \text{for } n \text{ even.}$$

The topology on $\mathbb{T}\text{-mod}(\text{Set})_c^{\text{op}}$ induced by these axioms is rigid: Given $M = (\cdots \rightarrow A_1 \rightarrow A_0) \in \mathbb{T}\text{-mod}(\text{Set})_c$ and an element $y \in A_n$ not in the image of f_n , the cosieve S_y is generated by the single model homomorphism

$$\begin{array}{ccccccc} \cdots & \rightarrow & A_{n+2} & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \rightarrow & \cdots & \rightarrow & A_0 \\ & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \cdots & \rightarrow & A_{n+2} & \rightarrow & A_{n+1} \sqcup \{x\} & \rightarrow & A_n & \rightarrow & \cdots & \rightarrow & A_0, \end{array}$$

where we have freely added a preimage x of y . After finitely many steps, we have covered M by a compact \mathbb{T} -model that satisfies \mathbb{Q}_1 .

However, denoting by \mathbb{Q}_2 the quotient extension requiring f_n to be surjective for n odd, which is a rigid-topology quotient for the same reason, the theory

$$\mathbb{T}' := \mathbb{T} + \mathbb{Q}_1 + \mathbb{Q}_2$$

is *not* of presheaf type. To see this, we show that $\mathbb{T}'\text{-mod}(\text{Set})$ is not finitely accessible. In fact, we can show that the only compact object of $\mathbb{T}'\text{-mod}(\text{Set})$ is the empty model $(\cdots \rightarrow \emptyset \rightarrow \emptyset)$, by considering the following sequence of inclusions.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \{0\} & \longrightarrow & \{0\} & \longrightarrow & \{0\} & \longrightarrow & \{0\} & = & M^0 \\ & & & & & & & & & & \downarrow \\ \cdots & \longrightarrow & \{0, 1\} & \longrightarrow & \{0, 1\} & \longrightarrow & \{0, 1\} & \xrightarrow{1 \mapsto 0} & \{0\} & = & M^1 \\ & & & & & & & & & & \downarrow \\ \cdots & \longrightarrow & \{0, 1, 2\} & \longrightarrow & \{0, 1, 2\} & \xrightarrow{2 \mapsto 1} & \{0, 1\} & \xrightarrow{1 \mapsto 0} & \{0\} & = & M^2 \\ & & & & & & & & & & \downarrow \\ & & & & \vdots & & & & & & \vdots \end{array}$$

The colimit of this diagram is a model M^ω with $A_n^\omega = \{0, \dots, n\}$. If $M = (\cdots \rightarrow A_1 \rightarrow A_0)$ is any nonempty model, then A_n is nonempty for all n , so the model homomorphism $M \rightarrow M^\omega$ sending any $x \in A_n$ to $n \in A_n^\omega$ does not factor through any of the models in the diagram, showing that M is not compact.

Finally, it seems appropriate to show that the theory of local rings, which played an important role in Subsection ?? and will do so again in Section 5, is not of presheaf type. Equivalently, this shows that the big Zariski topos $(\text{Spec } \mathbb{Z})_{\text{Zar}_{\text{fp}}}$ is not equivalent to a presheaf topos.

Proposition 4.5.6. *The theory of local rings, $\text{Ring} + (\text{loc})$ (see Definition 3.7.1), is not of presheaf type.*

Proof. The argument given here resembles the one given in [4, Section 9.4] showing that the usual geometric theory of fields is not of presheaf type. Assume that the theory of local rings is of presheaf type, that is, it is classified by $[\text{locRing}_c, \text{Set}]$ with the tautological local ring as universal model. We aim to reveal a subterminal object of $[\text{locRing}_c, \text{Set}]$ which is not definable by a closed geometric formula. Note that every local ring A is a $\mathbb{Z}_{(p)}$ -algebra for some prime number p , since at most one prime number can be non-invertible as an element of A . This means that the join of the subterminal objects

$$U_p := \text{Hom}(\mathbb{Z}_{(p)}, -) : \text{locRing}_c \rightarrow \text{Set}$$

is $\bigvee_p U_p = 1$. Now, if there were geometric formulas ϕ_p such that $\llbracket \phi_p \rrbracket = U_p$ (i.e. ϕ_p expresses that all prime numbers except p are invertible) for all p , then the disjunction $\bigvee_p \phi_p$, being valid in the universal model, would be provable in the theory of local rings. But since this is a coherent theory, a finite disjunction $\bigvee_{p \in P} \phi_p$ with P a finite set of primes would also have to be provable by [4, Theorem 10.8.6 (iii)]. This is not the case, as $\bigvee_{p \in P} \phi_p$ is falsified by the local ring $\mathbb{Z}_{(p')}$ for any $p' \notin P$. \square

5 Syntactic presentations for crystalline toposes

5.1 Introduction

In this section, we determine a syntactic presentation of the big crystalline toposes used for studying crystalline cohomology in algebraic geometry. This fulfills a promise made by Wraith in 1979 in [10, p. 743], where he writes: “It is my belief that a great many of the toposes occurring in algebraic geometry can be conveniently described in terms of the theories they classify. This is certainly so in the case of étale and crystalline toposes, for example.” This optimistic statement is however not accompanied by any hint of what the classified theory might be, and no answer has been given since.

The reason why the question is not as simple as for the big Zariski topos, for which the answer has been known since the infancy of the field of classifying toposes, is that much more data is involved in its construction. To wit, the crystalline topos depends on two schemes, a scheme morphism between them, and a certain structure known as a divided powers structure, or PD structure on one of them.

In the affine case, a PD structure on a ring A of positive characteristic p assigns to an element $a \in A$ another element of A , acting as a replacement for the expression

$$\frac{1}{n}a^n,$$

which can never be taken literally when p divides n . The intuition why such a structure might be useful for cohomological techniques is that in positive characteristic p , taking the derivative of a polynomial, or a power series,

$$t^n \mapsto nt^{n-1},$$

is never a surjective map, as there is no integral of t^{p-1} , and this leads to a failure of the Poincaré Lemma. But if divided powers of t are available, then the integral does exist.

For us, this means that the syntactic presentation of the crystalline topos will involve two rings and a geometric theory formulation of a PD structure. To handle this somewhat complex theory, we make heavy use of the techniques around theories of presheaf type developed in Section 4.

5.2 Background on divided power rings

Let us collect a number of properties and constructions of divided power rings that will be relevant. A *PD structure* or *divided powers structure* on an ideal I of a ring A (always commutative and unitary) is a family of functions

$$\gamma_n : I \rightarrow I, \quad n \geq 1$$

such that $x^n = n!\gamma_n(x)$ for every $x \in I$, compatible with natural operations on the ideal I and with each other, as specified in Definition 5.3.1 below. For example, the zero ideal (0) of any ring carries the trivial PD structure $\gamma_n(0) = 0$, while any ideal of a \mathbb{Q} -algebra A admits a unique PD structure given by $\gamma_n(a) = \frac{a^n}{n!}$. Slightly more interestingly, the

maximal ideal (p) of $\mathbb{Z}_{(p)}$ admits a unique PD structure, as the prime factor p occurs at most $n - 1$ times in $n!$, so $\frac{p^n}{n!} \in (p) \subseteq \mathbb{Z}_{(p)}$ for all $n \geq 1$.

An ideal equipped with a PD structure is called a *PD ideal* and a ring equipped with a PD ideal, (that is, a model in Set of the theory $\text{Ring} + \text{Ideal} + \text{PD}$, see Definition 5.3.1 below), is a *PD ring*. This also defines the homomorphisms of PD rings. We sometimes write a PD ring (A, I, γ) as $(\gamma \curvearrowright I \triangleleft A)$.

Not every ideal $J \subseteq I$ contained in the PD ideal of a PD ring (A, I, γ) is closed under the γ_n . But we can always close it up in one step, that is, the ideal generated by $\gamma_n(a)$ for $n \geq 1$ and $a \in J$ is a sub-PD-ideal of I , and it actually suffices to take $\gamma_n(a)$ for a in any generating set of the ideal J . More generally, for any ideal J of A , we let \bar{J} be the ideal generated by J and $\{\gamma_n(a) \mid n \geq 1, a \in J \cap I\}$ and call it the *PD saturation* of the ideal J (in the PD ring (A, I, γ)). There is an induced PD structure on the ideal $I/(J \cap I)$ of the ring A/J if and only if J is PD saturated ($J = \bar{J}$) [7, Tag 07H2], and this constitutes a bijective correspondence between PD saturated ideals of A and isomorphism classes of maps of PD rings $(A, I, \gamma) \rightarrow (A', I', \gamma')$ such that both $A \rightarrow A'$ and $I \rightarrow I'$ are surjective.

For a PD ring (A, I, γ) and an element $a \in A$, there is a unique PD structure γ_a on the ideal I_a of $A_a = A[a^{-1}]$ such that $A \rightarrow A_a$ is a homomorphism of PD rings, namely $\gamma_{a,n}(\frac{a'}{a^k}) = \frac{\gamma_n(a')}{a^{nk}}$. The new PD ring has the expected universal property: a homomorphism $(\gamma_a \curvearrowright I_a \triangleleft A_a) \rightarrow (\gamma' \curvearrowright I' \triangleleft A')$ is the same as one $(\gamma \curvearrowright I \triangleleft A) \rightarrow (\gamma' \curvearrowright I' \triangleleft A')$ that makes a invertible in A' .

The *PD envelope* construction [7, Tag 07H8] is a left adjoint functor for the forgetful functor

$$(\text{Ring} + \text{Ideal} + \text{PD})\text{-mod}(\text{Set}) \rightarrow (\text{Ring} + \text{Ideal})\text{-mod}(\text{Set}),$$

and more generally, for a fixed PD ring (K, I_K, γ_K) , a left adjoint for the forgetful functor

$$\begin{array}{ccc} (K\text{-Alg} + \text{Ideal}_{I_K} + \text{PD}_{\gamma_K})\text{-mod}(\text{Set}) & \longrightarrow & (K\text{-Alg} + \text{Ideal}_{I_K})\text{-mod}(\text{Set}) \\ \parallel & & \parallel \\ (K, I_K, \gamma_K)/(\text{Ring} + \text{Ideal} + \text{PD})\text{-mod}(\text{Set}) & & (K, I_K)/(\text{Ring} + \text{Ideal})\text{-mod}(\text{Set}). \end{array}$$

Forming the PD envelope (A', I', γ') of (A, I) (possibly over some (K, I_K, γ_K)) not only “enlarges” the ideal I as necessary but also the ring A . However, there is an isomorphism $A/I \cong A'/I'$, induced by the unit of the adjunction $\eta_{(A,I)} : (A, I) \rightarrow (A', I')$.

The polynomial ring $A[X]$ over a PD ring $A = (A, I, \gamma)$ inherits a PD structure on the ideal $I[X]$ (generated by the elements of I), such that PD maps $A[X] \rightarrow A'$ over (A, I, γ) correspond to elements of A' . The same is true for an arbitrary set of polynomial variables instead of a single one. But there is a separate notion of *PD polynomial algebra* [7, Tag 07H4], denoted $A\langle X \rangle$, which is freely generated as an A -module by 1 and the divided powers $\gamma_n(X)$ of X (instead of the ordinary powers X^n). The PD ideal of $A\langle X \rangle$ is generated by the elements of I and the $\gamma_n(X)$, and PD maps $A\langle X \rangle \rightarrow A'$ over (A, I, γ) correspond to elements of the PD ideal I' of A' . Equivalently, $A\langle X \rangle$ is the PD envelope of the ideal $I[X] + (X)$ in $A[X]$ over (A, I, γ) . Again, an arbitrary set of variables works just as well.

5.3 Relevant geometric theories

Here, we define the various theories and theory extensions from which the syntactic presentation of the crystalline topos will be built. We do this right away, since it will also be of some use in the definition of the crystalline topos itself. In particular, by giving our formal definition of a PD structure in the form of an extension of geometric theories, it can immediately be used to define PD structures on schemes as well.

Definition 5.3.1.

- (i) We denote Ideal the extension of the theory Ring (see Definition 3.7.1) consisting of a relation symbol $I \subseteq A$ with these axioms:

$$0 \in I, \quad x \in I \wedge y \in I \vdash_{x,y:A} x + y \in I, \quad x \in I \vdash_{\lambda,x:A} \lambda x \in I.$$

- (ii) In the theory Ring + Ideal, we can require the ideal to be a nil ideal by adding the axiom:

$$(\text{nil}) := \quad x \in I \vdash_{x:A} \bigvee_{n \in \mathbb{N}} (x^n = 0).$$

- (iii) We denote PD-Ideal the following extension of the theory Ring. First introduce a sort S_I , a function symbol $\iota : S_I \rightarrow A$ with

$$\iota(x) = \iota(y) \vdash_{x,y:S_I} x = y$$

and function symbols $0 : I$, $+$: $I \times I \rightarrow I$ and \cdot : $A \times I \rightarrow I$ together with equational axioms stating that ι is an A -module homomorphism. Then add function symbols $\gamma_n : S_I \rightarrow S_I$ for all $n \geq 1$ and these equational axioms (see [7, Tag 07GL]):

- $\gamma_1(x) = x$
- $\gamma_n(x + y) = \gamma_n(x) + \gamma_n(y) + \sum_{i+j=n, i,j \geq 1} \gamma_i(x)\gamma_j(y)$
- $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$
- $\iota(\gamma_m(x))\gamma_n(x) = \binom{m+n}{m} \gamma_{m+n}(x)$
- $\gamma_m(\gamma_n(x)) = \frac{(mn)!}{m!(n!)^m} \gamma_{mn}(x)$

- (iv) To be able to talk about a PD structure on an existing ideal, we define the extension PD of Ring + Ideal to be PD-Ideal plus the axiom

$$\exists x : S_I. \iota(x) = y \dashv\vdash_{y:A} y \in I.$$

Note that Ideal+PD and PD-Ideal are Morita equivalent extensions of the theory Ring, since the axioms in Ideal are redundant here, and the relation symbol I is definable by the formula $\exists x : S_I. \iota(x) = y$. We prefer to write Ring + Ideal + PD because we think of the divided powers as an additional structure on an ideal. Also note that PD is equivalent to a localic extension of Ring + Ideal, as the divided power structure could alternatively be implemented as relation symbols $\tilde{\gamma}_n \subseteq A \times A$ expressing partial functions from A to A . The main purpose of the sort S_I above is to be able to write the axioms in a more readable equational style.

Definition 5.3.2.

- (i) Similarly to the extension $K\text{-AlgStr}$ of Ring from Definition 3.7.1, for a K -algebra R we denote $R\text{-AlgStr}_K$ the extension of $K\text{-Alg}$ adding an R -algebra structure compatible with the given K -algebra structure. If necessary, we write $R\text{-AlgStr}_K(A)$ to indicate the sort A to which the R -algebra structure is added.
- (ii) For a ring K and a K -algebra R , we denote $K\text{-Alg-}R\text{-Alg}$ the theory $K\text{-Alg} + R\text{-Alg}$ extended by a function symbol $f : A \rightarrow B$ (where A is the K -algebra and B is the R -algebra) and equational axioms expressing that f is a K -algebra homomorphism. A model can be pictured as

$$\begin{array}{ccc} K & \longrightarrow & R \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B. \end{array}$$

When we treat $K\text{-Alg-}R\text{-Alg}$ as if it contained $K\text{-Alg} + \text{Ideal}$, we intend to use the kernel of f as the ideal, that is, $x \in I$ is defined as $f(x) = 0$.

- (iii) We set $K\text{-Alg-}R\text{-Quot} := K\text{-Alg-}R\text{-Alg} + (\text{surj})$, where

$$(\text{surj}) := \quad \top \vdash_{y:B} \exists x:A. f(x) = y.$$

- (iv) For a ring K and an ideal $I_K \subseteq K$, we denote Ideal_{I_K} the extension of $K\text{-Alg}$ consisting of the extension Ideal and the additional axioms $c_\lambda \in I$ for all $\lambda \in I_K$.
- (v) For a PD ring (K, I_K, γ_K) , we denote PD_{γ_K} the extension of $K\text{-Alg} + \text{Ideal}_{I_K}$ consisting of the extension PD and the additional axioms (for every $\lambda \in I_K$)

$$\iota(x) = c_\lambda \vdash_{x:S_I} \iota(\gamma_n(x)) = c_{\gamma_{K,n}(\lambda)}.$$

We can write $K\text{-Alg-}R\text{-Alg} + \text{PD}_{\gamma_K}$ if I_K vanishes in R , since $K\text{-Alg-}R\text{-Alg}$ then proves $f(c_\lambda) = 0$ for $\lambda \in I_K$.

Similarly to the economical version of the theory $K\text{-Alg}$ mentioned in Remark 3.7.2, we can use a presentation $R = K[X_i]/(r_j)$ of R as a K -algebra to implement the R -algebra structure of B in the theory $K\text{-Alg-}R\text{-Alg}$. That is, $K\text{-Alg-}R\text{-Alg}$ is equivalent to the extension of $K\text{-Alg-}K\text{-Alg}$ where we only add constant symbols $c_i : B$ and axioms $r_j(c_i) = 0$, using the available K -algebra structure to interpret $r_j(c_i)$ as a term of sort B . This will be used in Propositions 5.5.1 and 5.7.3.

Remark 5.3.3. The categories of models $K\text{-Alg-mod}(\text{Set})$, $(K\text{-Alg} + \text{Ideal}_{I_K})\text{-mod}(\text{Set})$ and $(K\text{-Alg} + \text{Ideal}_{I_K} + \text{PD}_{\gamma_K})\text{-mod}(\text{Set})$ are equivalent to the slice categories $K/(\text{Ring-mod}(\text{Set}))$, $(K, I_K)/((\text{Ring} + \text{Ideal})\text{-mod}(\text{Set}))$ and $(K, I_K, \gamma_K)/((\text{Ring} + \text{Ideal} + \text{PD})\text{-mod}(\text{Set}))$, respectively. Even $(K\text{-Alg-}R\text{-Alg})\text{-mod}(\text{Set})$ is equivalent to $(K \rightarrow R)/(\text{Ring}^{\rightarrow}\text{-mod}(\text{Set}))$, if we use a theory $\text{Ring}^{\rightarrow}$ of ring homomorphisms. We could in fact define these theories by a general construction that turns a geometric theory \mathbb{T} and a Set -model M of \mathbb{T} into a theory of “ M -algebras”. This is the same construction that is used by

Blechsmidt in [3, Definition 3.1], who however applies it internally in the classifying topos of \mathbb{T} , to the universal model of \mathbb{T} instead of a Set-based model. But for our present goals, we need to be able to combine the above theories and extensions in flexible ways, so that such a formulation does not help us much.

5.4 Definition of the big crystalline topos

Before we can define the crystalline topos, we need to introduce a number of notions involving PD structures on schemes. Recall that a closed embedding $U \rightarrow T$ defined by a quasi-coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_T$ is a *thickening* (in the terminology of [7, Tag 04EX]), that is, $U \rightarrow T$ is a homeomorphism, if and only if every local section of \mathcal{I} is locally nilpotent, which in turn means simply that the internal ideal \mathcal{I} of the internal ring \mathcal{O}_T in $\text{Sh}(T)$ satisfies the axiom (nil).

Definition 5.4.1 (see [7, Tag 07I1]).

- (i) A *PD scheme* is a scheme S together with a quasi-coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_S$ and a PD structure on \mathcal{I} , that is, a model extension along PD of the (Ring + Ideal)-model $(\mathcal{I} \triangleleft \mathcal{O}_S)$ in $\text{Sh}(S)$. A *morphism* of PD schemes $(S, \mathcal{I}, \gamma) \rightarrow (S', \mathcal{I}', \gamma')$ is a morphism of schemes $f : S \rightarrow S'$ such that $f^\# : f^{-1}\mathcal{O}_{S'} \rightarrow \mathcal{O}_S$ is a (Ring + Ideal + PD)-model homomorphism. In particular, it induces a morphism between the closed subschemes of S and S' defined by \mathcal{I} respectively \mathcal{I}' ,

$$\begin{array}{ccc} S' & \longleftarrow & V(\mathcal{I}') \\ \uparrow & & \uparrow \\ S & \longleftarrow & V(\mathcal{I}). \end{array}$$

- (ii) A *PD thickening* is a PD scheme (T, \mathcal{I}, γ) such that the closed embedding $V(\mathcal{I}) \rightarrow T$ is a thickening, that is, the (Ring + Ideal + PD)-model $(\mathcal{O}_T, \mathcal{I}, \gamma)$ in $\text{Sh}(T)$ fulfills the axiom (nil). PD thickenings are sometimes denoted (U, T, γ) , where $U = V(\mathcal{I})$.
- (iii) If $(S, \mathcal{I}_S, \gamma_S)$ is a PD scheme and X is a scheme over the closed subscheme $S_0 := V(\mathcal{I}_S)$ of S , then a *PD thickening over S and X* is a PD thickening (T, \mathcal{I}, γ) over $(S, \mathcal{I}_S, \gamma_S)$ together with a morphism of schemes $V(\mathcal{I}) \rightarrow X$ over S .

$$\begin{array}{ccccc} S & \longleftarrow & S_0 & \longleftarrow & X \\ \uparrow & & & & \uparrow \\ T & \longleftarrow & & & V(\mathcal{I}) \end{array}$$

A *morphism* of PD thickenings over S and X is a morphism of PD schemes over S such that the induced morphism $V(\mathcal{I}) \rightarrow V(\mathcal{I}')$ respects the structure morphisms to X .

The PD thickenings over S and X will be the objects of the crystalline site of X/S . Most important to us is that if S and X are affine and we also require T to be affine, then we can nicely describe the resulting category by a geometric theory.

Lemma 5.4.2. *The full subcategory of the category of PD thickenings where the underlying scheme T is affine is equivalent to*

$$(\text{Ring} + \text{Ideal} + \text{PD} + (\text{nil}))\text{-mod}(\text{Set})^{\text{op}}.$$

For a fixed PD ring (K, I_K, γ_K) and a K/I_K -algebra R , the full subcategory of the category of PD thickenings over $S = \text{Spec}(K)$ and $X = \text{Spec}(R)$ where the underlying scheme T is affine is equivalent to

$$(K\text{-Alg-}R\text{-Quot} + \text{PD}_{\gamma_K} + (\text{nil}))\text{-mod}(\text{Set})^{\text{op}}.$$

Proof. The category $(\text{Ring} + \text{Ideal})\text{-mod}(\text{Set})$ is of course dual to the category of affine schemes equipped with a quasi-coherent ideal sheaf, via $(I \triangleleft A) \mapsto (\text{Spec } A, \tilde{I})$. One can check that the closed embedding $\text{Spec}(A/I) \rightarrow \text{Spec } A$ is a thickening if and only if every element of I is nilpotent. Similarly, a PD structure on I uniquely determines a PD structure on \tilde{I} , and morphisms respecting one correspond to morphisms respecting the other [1, p. 3.18].

For the second part, the theory extension PD_{γ_K} (requiring a PD structure on the kernel of $f : A \rightarrow B$ extending γ_K) makes sense because the PD ideal I_K vanishes in R , so the theory $K\text{-Alg-}R\text{-Quot}$ proves $f(c_\lambda) = 0$ for all $\lambda \in I_K$. The rest follows from the first part. \square

If (T, \mathcal{I}, γ) is a PD thickening and $T' \subseteq T$ is an open subscheme, then $(T', \mathcal{I}|_{T'}, \gamma|_{T'})$ is again a PD thickening (as restricting to T' is pulling back along the geometric morphism $\text{Sh}(T') \rightarrow \text{Sh}(T)$) and there is a canonical inclusion morphism $(T', \mathcal{I}|_{T'}, \gamma|_{T'}) \rightarrow (T, \mathcal{I}, \gamma)$. This allows us to define a Zariski topology on the category of PD thickenings over S and X .

Definition 5.4.3. Let $(S, \mathcal{I}_S, \gamma_S)$ be a PD scheme and let X be a scheme over $S_0 := V(\mathcal{I}_S)$. The *big crystalline site* $\text{Cris}(X/S)$ is the category of PD thickenings over S and X , endowed with the Zariski topology, that is, a sieve on (T, \mathcal{I}, γ) is covering if and only if it contains all the arrows $(T_i, \mathcal{I}|_{T_i}, \gamma|_{T_i}) \rightarrow (T, \mathcal{I}, \gamma)$ for some open cover $T = \bigcup_i T_i$ of T .

If we want to consider the topos $\text{Sh}(\text{Cris}(X/S))$ of sheaves on the crystalline site, there remains an issue of size. There is no way to talk about the collection of presheaves on a big category such as $\text{Cris}(X/S)$ (or generally, about the collection of functions between two proper classes) in ZFC set theory. If the category in question is essentially small, then one can circumvent this by choosing a skeleton. More generally, if J is a Grothendieck topology on a category \mathcal{C} such that there is a (small) dense set $S \subseteq \mathcal{C}$ of objects of \mathcal{C} , then one can write

$$\text{Sh}(\mathcal{C}, J) := \text{Sh}(S, J|_S)$$

(where $J|_S$ is the greatest topology on S such that $S \hookrightarrow \mathcal{C}$ preserves covers). This is justified because if $S' \subseteq \mathcal{C}$ is another dense set of objects, then by the Comparison Lemma, there are canonical equivalences

$$\text{Sh}(S, J|_S) \simeq \text{Sh}(S \cup S', J|_{S \cup S'}) \simeq \text{Sh}(S', J|_{S'}).$$

However, the site $\text{Cris}(X/S)$ does not admit a dense set of objects. To define a crystalline topos, a choice of some class $\mathcal{C} \subseteq \text{Cris}(X/S)$ which does admit a dense subset is therefore necessary. We leave this choice as an explicit parameter to the definition for now, but we will specialize to a particular class \mathcal{C} in Definition 5.7.1.

To avoid an ambiguity in the topology on the resulting site, we require \mathcal{C} to be closed under taking open subschemes $T' \subseteq T$ (with the induced PD structure on T'). While there is always the induced topology $J_{\text{Zar}}|_{\mathcal{C}}$, for which a sieve on $T \in \mathcal{C}$ is covering if and only if it generates a covering sieve in $\text{Cris}(X/S)$, if \mathcal{C} does not contain all open subschemes of T , then there is no guarantee that such a sieve actually contains a “distinguished” Zariski cover by open subschemes. With this assumption, the two conditions become equivalent.

Definition 5.4.4. Let S, X be as before and let $\mathcal{C} \subseteq \text{Cris}(X/S)$ be a class of objects closed under taking open subschemes. Then we denote $\text{Cris}_{\mathcal{C}}(X/S)$ the full subcategory on these objects endowed with the Zariski topology as above. And the topos

$$(X/S)_{\text{Cris}_{\mathcal{C}}} := \text{Sh}(\text{Cris}_{\mathcal{C}}(X/S))$$

is the *big crystalline topos* of X over $(S, \mathcal{I}_S, \gamma_S)$ (defined using the objects in \mathcal{C}).

There is a structure sheaf on $\text{Cris}_{\mathcal{C}}(X/S)$ (see [7, Tag 07IN]),

$$\mathcal{O} = \mathcal{O}_{(X/S)_{\text{Cris}_{\mathcal{C}}}} : \text{Cris}_{\mathcal{C}}(X/S)^{\text{op}} \rightarrow \text{Set}, \quad (T, \mathcal{I}, \gamma) \mapsto \mathcal{O}_T(T).$$

As in the case of the Zariski topos, this sheaf carries a ring structure. But here, there is more. We have a second ring object

$$\mathcal{O}' = \mathcal{O}'_{(X/S)_{\text{Cris}_{\mathcal{C}}}} : \text{Cris}_{\mathcal{C}}(X/S)^{\text{op}} \rightarrow \text{Set}, \quad (T, \mathcal{I}, \gamma) \mapsto \mathcal{O}_U(U) = \mathcal{O}_{V(\mathcal{I})}(V(\mathcal{I})),$$

sometimes denoted \mathcal{O}_X . Furthermore, there is a surjective ring homomorphism $\mathcal{O} \rightarrow \mathcal{O}'$, and its kernel

$$\mathcal{J} = \mathcal{J}_{(X/S)_{\text{Cris}_{\mathcal{C}}}} : \text{Cris}_{\mathcal{C}}(X/S)^{\text{op}} \rightarrow \text{Set}, \quad (T, \mathcal{I}, \gamma) \mapsto \mathcal{I}(T)$$

carries a canonical PD structure. In summary, we have a model

$$(\mathcal{O}, \mathcal{J}) \in (\text{Ring} + \text{Ideal} + \text{PD})\text{-mod}((X/S)_{\text{Cris}_{\mathcal{C}}}).$$

An extension of this model (along a localic theory extension) will turn out to be the universal model of a theory classified by $(X/S)_{\text{Cris}_{\mathcal{C}}}$, at least for an appropriate choice of \mathcal{C} .

The Zariski topology on $\text{Cris}(X/S)$ (or $\text{Cris}_{\mathcal{C}}(X/S)$) in particular ensures that every object can be covered by objects (T, \mathcal{I}, γ) with T affine, in other words, the affine objects are dense. Thus we can use the Comparison Lemma to obtain an equivalent site consisting of affine objects, which we have already described as a category in Lemma 5.4.2. The following lemma complements this with a description of the topology on the new site.

Lemma 5.4.5. *Let (K, I_K, γ_K) be a PD ring and let R be a K/I_K -algebra. The topology induced via the Comparison Lemma on $(K\text{-Alg-}R\text{-Quot} + \text{PD}_{\gamma_K} + (\text{nil}))\text{-mod}(\text{Set})^{\text{op}}$ regarded as a (dense) full subcategory of $\text{Cris}(\text{Spec } R/\text{Spec } K)$ is the following: A cosieve on an object*

$$(\gamma \curvearrowright I \triangleleft A \twoheadrightarrow B)$$

is covering if and only if it contains the canonical arrows to

$$(\gamma_{a_i} \curvearrowright I_{a_i} \triangleleft A_{a_i} \twoheadrightarrow B_{a_i})$$

for some finite family of elements $a_i \in A$ with $(a_1, \dots, a_n) = (1)$.

Proof. We first note that the model $(\gamma_{a_i} \curvearrowright I_{a_i} \triangleleft A_{a_i} \twoheadrightarrow B_{a_i})$ is well-defined — the ring map $A_{a_i} \rightarrow B_{a_i}$ is surjective and has kernel I_{a_i} since localization is exact, and I_{a_i} is still a nil ideal — and corresponds to the open subscheme $D(a_i)$ of the PD scheme $\text{Spec } A_{a_i}$. A cosieve on $(\gamma \curvearrowright I \triangleleft A \twoheadrightarrow B)$ is covering for the induced topology if it generates a covering sieve in $\text{Cris}(\text{Spec } R/\text{Spec } K)$. But such a sieve contains (the morphisms of PD thickenings over $\text{Spec } K$ and $\text{Spec } R$ induced by) an open cover of $\text{Spec } A$ if and only if it contains a cover by standard opens $D(a_i)$, which in turn cover $\text{Spec } A$ if and only if $(a_1, \dots, a_n) = (1)$. \square

5.5 Preliminary presheaf type results

We saw in Lemma 5.4.2 that the crystalline site of affine schemes $S = \text{Spec } K$, $X = \text{Spec } R$ is closely tied to the theory

$$K\text{-Alg-}R\text{-Quot} + \text{PD}_{\gamma_K} + (\text{nil}).$$

Our strategy for proving a classification result on the crystalline topos is to show, under appropriate assumptions, that this theory is of presheaf type, then choose the class of objects of the crystalline site in such a way that the affine objects turn out to be exactly the compact models of the theory, and finally add the quotient extension (loc) to produce the Zariski topology.

Right now, we want to approach the presheaf type part by looking at somewhat simpler theories. While $K\text{-Alg-}R\text{-Alg}$ is of presheaf type simply because it is an algebraic theory, the situation already becomes more interesting for $K\text{-Alg-}R\text{-Quot}$, as the axiom (surj) is not cartesian. We also include the extension PD but assume the PD structure on K to be trivial and forget about (nil) for now.

The following two propositions can also serve as an illustration of the different ways in which one might try to decompose the same theory into parts to show that it is of presheaf type, and how this can lead to results of different strengths.

Proposition 5.5.1. *Let R be a finitely presented K -algebra. Then the theory*

$$K\text{-Alg-}R\text{-Quot} + \text{PD}$$

is of presheaf type.

Proof. The cartesian theory $K\text{-Alg} + \text{Ideal} + \text{PD}$ is Morita-equivalent to $K\text{-Alg} - K\text{-Quot} + \text{PD}$ by introducing a sort B for the quotient ring A/I , similarly to Example 2.4.2. So the latter theory is of presheaf type and we are only missing the R -algebra structure on the K -algebra B . Since $R \cong K[X_1, \dots, X_n]/(r_1, \dots, r_m)$ is a finitely presented K -algebra, the extension $R\text{-AlgStr}_K(B)$ is equivalent to finitely many constant symbols $c_i : B$, $i = 1, \dots, n$, and finitely many axioms in the empty context $\top \vdash_{\square} r_j(\vec{c}) = 0$, $j = 1, \dots, m$. Thus we are done by Proposition 4.4.5 and Corollary 4.4.2. \square

We can reduce the assumption on R to finite type instead of finite presentation if we handle the axiom (surj) in a different way. For this, we need a lemma.

Lemma 5.5.2. *Let $M = (\gamma \curvearrowright \mathfrak{a} \triangleleft A \xrightarrow{f} B) \in \mathcal{C} = (K\text{-Alg} - R\text{-Alg} + \text{PD})\text{-mod}(\text{Set})$ and $b \in B$. Then there is a universal triple (M', a', g) , where $M' = (\gamma' \curvearrowright \mathfrak{a}' \triangleleft A' \xrightarrow{f'} B') \in \mathcal{C}$, $a' \in A'$ and $g = (g_A, g_B) : M \rightarrow M'$, such that $f'(a') = g_B(b)$; that is, the following diagram in $[\mathcal{C}, \text{Set}]$ is a pullback:*

$$\begin{array}{ccc} \text{Hom}(M', -) & \xrightarrow{g^*} & \text{Hom}(M, -) \\ a' \downarrow & \lrcorner & \downarrow b \\ \llbracket A \rrbracket_- & \xrightarrow{\llbracket f \rrbracket_-} & \llbracket B \rrbracket_- \end{array}$$

Furthermore, $g_B : B \rightarrow B'$ is an isomorphism.

Proof. Consider the A -algebra homomorphism $\tilde{f} : \tilde{A} := A[X] \rightarrow B$ with $\tilde{f}(X) = b$ and set $\tilde{\mathfrak{a}} := \ker \tilde{f}$. Let $\gamma' \curvearrowright \mathfrak{a}' \triangleleft A'$ be the PD envelope (over γ) of the $(\mathfrak{a} \triangleleft A)$ -algebra $\tilde{\mathfrak{a}} \triangleleft \tilde{A}$. In particular, we have $\tilde{A}/\tilde{\mathfrak{a}} \cong A'/\mathfrak{a}'$ (as A -algebras) via the unit $\eta_{\tilde{\mathfrak{a}} \triangleleft \tilde{A}} : (\tilde{\mathfrak{a}} \triangleleft \tilde{A}) \rightarrow (\mathfrak{a}' \triangleleft A')$ of the adjunction, so that \tilde{f} induces an A -algebra homomorphism $f' : A' \rightarrow B$ with $\ker f' = \mathfrak{a}'$. We have thus defined a model $M' \in \mathcal{C}$ and an arrow $g : M \rightarrow M'$, where $g_A : A \rightarrow A'$ is just the homomorphism of PD-rings from the PD envelope construction and $g_B = \text{id}_B$. Setting $a' := \eta_{\tilde{\mathfrak{a}} \triangleleft \tilde{A}}(X)$ satisfies $f'(a') = b$.

To check the universal property, let $h : M \rightarrow M'' = (\gamma'' \curvearrowright \mathfrak{a}'' \triangleleft A'' \xrightarrow{f''} B'')$ be given. An element $a'' \in A'' = \llbracket A \rrbracket_{M''}$ with $f''(a'') = h_B(b)$ is the same thing as an A -algebra homomorphism $\tilde{A} \rightarrow A''$ fitting in the square

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\tilde{f}} & B \\ \downarrow & & \downarrow h_B \\ A'' & \xrightarrow{f''} & B'' \end{array}$$

But then it is automatic that $\tilde{\mathfrak{a}}$ is sent to \mathfrak{a}'' , so there is a unique homomorphism of PD-rings $(\gamma' \curvearrowright \mathfrak{a}' \triangleleft A') \rightarrow (\gamma'' \curvearrowright \mathfrak{a}'' \triangleleft A'')$ over A sending a' to a'' , and it makes the square

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B \\ \downarrow & & \downarrow h_B \\ A'' & \xrightarrow{f''} & B'' \end{array}$$

commute, as required for a model homomorphism $M' \rightarrow M''$. \square

Proposition 5.5.3. *Let R be a finitely generated K -algebra. Then the theory*

$$K\text{-Alg-}R\text{-Quot} + \text{PD}$$

is of presheaf type.

Proof. The theory $K\text{-Alg-}R\text{-Alg} + \text{PD}$ (with a PD structure on the kernel of the K -algebra homomorphism $A \rightarrow B$) is of presheaf type because it is cartesian. We show that the missing axiom (surj) is a rigid-topology quotient.

Let a compact model

$$M = (\gamma \curvearrowright \mathfrak{a} \triangleleft A \xrightarrow{f} B)$$

of $K\text{-Alg-}R\text{-Alg} + \text{PD}$ and an element $b \in B$ be given. Consider the model homomorphism $g : M \rightarrow M'$ from Lemma 5.5.2. We first note that M' is also compact, since $\text{Hom}(M', -)$ is a finite limit of functors preserving filtered colimits. (For $\llbracket A \rrbracket_-$ and $\llbracket B \rrbracket_-$ preserving filtered colimits, see Lemma 4.2.2.) Secondly, a map $h : M \rightarrow M''$ factors through M' (not necessarily uniquely) if and only if h_B maps b to something in the image of f'' . This means that the cosieve on M generated by g (in the category $(K\text{-Alg-}R\text{-Alg} + \text{PD})\text{-mod}(\text{Set})_c$) is exactly the cosieve S_b as in Theorem 4.3.1 and thus M is $J_{(\text{surj})}$ -covered by M' . And lastly, the subset $f'(A') \subseteq B' = B$ contains of course both $f(A)$ and the element b .

Now, since M is compact, B is a finitely presented R -algebra. This follows from Lemma 4.2.3 by observing that the right adjoint of the forgetful functor in the following diagram preserves filtered colimits.

$$\begin{array}{ccc} (K\text{-Alg-}R\text{-Alg} + \text{PD})\text{-mod}(\text{Set}) & \longrightarrow & (R\text{-Alg})\text{-mod}(\text{Set}) \\ (\gamma \curvearrowright \mathfrak{a} \triangleleft A \rightarrow B) & \longmapsto & B \\ ((0) \triangleleft B' \xrightarrow{\cong} B') & \longleftarrow \perp & B' \end{array}$$

Thus, B is also a finitely generated K -algebra by assumption. But if $b_1, \dots, b_n \in B$ are K -algebra generators of B , then we can successively apply Lemma 5.5.2 to them to obtain a covering arrow $M \rightarrow M_n$ (using the transitivity of the topology $J_{(\text{surj})}$) such that $b_i \in f_n(A_n)$ for all i , implying that f_n is surjective. \square

Remark 5.5.4. Propositions 5.5.1 and 5.5.3 become wrong if we drop the finiteness assumption entirely. For example, take $K = \mathbb{Q}$, $R = \mathbb{C}$. The extension PD is an equivalence extension of $\mathbb{Q}\text{-Alg-}\mathbb{C}\text{-Quot}$ because the γ_n are definable by

$$\gamma_n(x) = y \dashv\vdash_{x,y:S_I} \frac{1}{n!} \iota(x)^n = \iota(y),$$

using the available \mathbb{Q} -algebra structure. And $\mathbb{Q}\text{-Alg-}\mathbb{C}\text{-Quot}$ has too few compact Set-based models to be of presheaf type, similarly to the situation in Example 4.5.2.

Namely, we can apply Lemma 4.2.3 to the pair of adjoint functors

$$\begin{array}{ccc} (\mathbb{Q}\text{-Alg-}\mathbb{C}\text{-Quot})\text{-mod}(\text{Set}) & \longrightarrow & (\mathbb{Q}\text{-Alg})\text{-mod}(\text{Set}) \\ (A \twoheadrightarrow B) & \longleftarrow & A \\ & \perp & \\ (A' \twoheadrightarrow 0) & \longleftarrow & A' \end{array}$$

to see that if $A \twoheadrightarrow B$ is a compact model of $\mathbb{Q}\text{-Alg-}\mathbb{C}\text{-Quot}$, then A is a finitely generated \mathbb{Q} -algebra. Then the \mathbb{C} -algebra B is also a finitely generated \mathbb{Q} -algebra, which implies $B = 0$ for cardinality reasons. But there are certainly non-compact models with $B \neq 0$ (for example, let A be freely generated by the elements of B), so the sequent

$$\top \vdash_{\square} 1_B = 0_B$$

can not be provable in $\mathbb{Q}\text{-Alg-}\mathbb{C}\text{-Quot}$, meaning that the compact Set -based models are not jointly conservative and therefore $\mathbb{Q}\text{-Alg-}\mathbb{C}\text{-Quot}$ is not of presheaf type (see [4, Theorem 6.1.1]).

We can now prove a “lazy version” of our classification result, where we simply assume the crystalline site to contain exactly the desired objects. More precisely, we even have to assume that a suitable class of objects $\mathcal{C} \subseteq \text{Cris}(X/S)$ exists, which is not trivial given the additional assumption that \mathcal{C} is closed under taking open subschemes. The axiom (nil) is made redundant here by an additional assumption on K .

Corollary 5.5.5. *Let K be a ring of nonzero characteristic, regard K as a PD ring with the trivial PD ideal (0), and let R be a finitely generated K -algebra. Assume there is a class $\mathcal{C} \subseteq \text{Cris}(\text{Spec } R/\text{Spec } K)$ closed under open subschemes, such that an affine object belongs to \mathcal{C} if and only if the corresponding model of $K\text{-Alg-}R\text{-Quot} + \text{PD} + (\text{nil})$ is compact. Then the topos $(\text{Spec } R/\text{Spec } K)_{\text{Cris}_{\mathcal{C}}}$ classifies the theory*

$$K\text{-Alg-}R\text{-Quot} + \text{PD} + (\text{nil}) + (\text{loc}).$$

Proof. If $n = 0$ in K for some $n \geq 1$, then for any PD ring (A, I, γ) over K and any $a \in I$, we have $a^n = n! \gamma_n(a) = 0$. This calculation can be carried out within the theory $K\text{-Alg-}R\text{-Quot} + \text{PD}$, so that the axiom (nil) is redundant. Thus by Proposition 5.5.3, $K\text{-Alg-}R\text{-Quot} + \text{PD} + (\text{nil})$ is of presheaf type. By our assumption on \mathcal{C} , the underlying category of the site of affine objects for $(\text{Spec } R/\text{Spec } K)_{\text{Cris}_{\mathcal{C}}}$ is

$$(K\text{-Alg-}R\text{-Quot} + \text{PD} + (\text{nil}))\text{-mod}(\text{Set})_c^{\text{op}},$$

the site for the presheaf topos classifying this theory. The only difference is the Zariski topology as described in Lemma 5.4.5. By Theorem 4.3.1, this is precisely the topology corresponding to the quotient extension (loc), as each arrow

$$(\gamma \curvearrowright I \triangleleft A \twoheadrightarrow B) \longrightarrow (\gamma_{a_i} \curvearrowright I_{a_i} \triangleleft A_{a_i} \twoheadrightarrow B_{a_i})$$

is the universal arrow that sends $a_i \in A$ to something invertible. \square

Remark 5.5.6. When adding (loc) to a theory like $K\text{-Alg-}R\text{-Alg}$, we mean to impose the axioms on the K -algebra A , not on the R -algebra B . However, in the presence of (surj) and (nil), the two options are in fact syntactically equivalent. This is because an elementary calculation using these axioms shows

$$\text{inv}(f(x)) \dashv\vdash_{x:A} \text{inv}(x).$$

5.6 Finiteness conditions for PD schemes

The set of objects that we include in the crystalline site will be determined by the compactness condition on models of the appropriate geometric theory.

Lemma 5.6.1. *Let (K, I_K, γ_K) be a PD ring. The compact models of $K\text{-Alg} + \text{Ideal}_{I_K} + \text{PD}_{\gamma_K}$ are those PD rings over K which are of the form*

$$K\langle X_1, \dots, X_m \rangle [Y_1, \dots, Y_n] / \overline{(r_1, \dots, r_k)}.$$

Proof. While the theory $\text{Ring} + \text{Ideal} + \text{PD}$ as defined in Definition 5.3.1 is cartesian, the equivalent theory $\text{Ring} + \text{PD-Ideal}$ is even a Horn theory. We can similarly construct a Horn theory equivalent to $K\text{-Alg} + \text{Ideal}_{I_K} + \text{PD}_{\gamma_K}$ by adding constants $\tilde{c}_\lambda : S_I$ in addition to $c_\lambda : A$ for every $\lambda \in I_K$, with $\iota(\tilde{c}_\lambda) = c_\lambda$.

By Lemma 4.2.4, the compact models of a Horn theory are those presented by some Horn formula in context. In our case, the model presented by the formula \top in the context $\vec{x} : S_I^m, \vec{y} : A^n$ is $K\langle \vec{X} \rangle [\vec{Y}]$. And since our theory has no relation symbols and the inclusion $\iota : S_I \rightarrow A$ is required to be injective, any atomic formula in this context is provably equivalent to an equality of two terms of sort A , and therefore also to one of the form $r = 0$, where r is a term representing an element of $K\langle \vec{X} \rangle [\vec{Y}]$. Thus we are done, as the universal arrow out of $K\langle \vec{X} \rangle [\vec{Y}]$ that kills r_1, \dots, r_n is $K\langle \vec{X} \rangle [\vec{Y}] \rightarrow K\langle \vec{X} \rangle [\vec{Y}] / \overline{(r)}$. \square

We now want to globalize this property of PD rings over (K, I, γ) to be able to apply it to morphisms of PD schemes. This parallels the treatment of morphisms of finite type and morphisms of finite presentation as for example in [7, Tag 01T0] and [7, Tag 01T0].

Definition 5.6.2. We say a homomorphism of PD rings is *of finite PD type* if it is of the form

$$(K, I_K, \gamma_K) \rightarrow K\langle \vec{X} \rangle [\vec{Y}] / \rho$$

for finite sets of variables $\vec{X} = X_1, \dots, X_n$ and $\vec{Y} = Y_1, \dots, Y_m$, where $\rho = \bar{\rho}$ is any PD saturated ideal of $K\langle \vec{X} \rangle [\vec{Y}]$. We say it is *of finite PD presentation* if it is of the form $(K, I_K, \gamma_K) \rightarrow K\langle \vec{X} \rangle [\vec{Y}] / \overline{(r)}$ for finitely many elements $r_1, \dots, r_k \in K\langle \vec{X} \rangle [\vec{Y}]$.

Lemma 5.6.3. *A homomorphism of PD rings $(K, I_K, \gamma_K) \rightarrow (A, I, \gamma)$ is of finite PD type if and only if there are elements $x_1, \dots, x_n \in I$ and $y_1, \dots, y_m \in A$ such that A is generated as a K -algebra by $\gamma_k(x_i)$ and y_i , and the ideal I is generated by the image of I_K and $\gamma_k(x_i)$.*

Proof. The choice of elements $x_i \in I$ and $y_j \in A$ corresponds to a homomorphism $f : K\langle \vec{X} \rangle[\vec{Y}] \rightarrow A$. The kernel of f is always a PD saturated ideal. This f is surjective if and only if A is generated as a K -algebra by $\gamma_k(x_i)$ and y_j . And in this case, the induced map between the PD ideals of $K\langle \vec{X} \rangle[\vec{Y}]$ and A is surjective if and only if I is generated by the image of I_K and the $\gamma_k(x_i)$. \square

Lemma 5.6.4. *Homomorphisms of finite PD presentation have the following properties.*

- (i) *If $R \rightarrow A \rightarrow A'$ are homomorphisms of PD rings with $R \rightarrow A$ and $R \rightarrow A'$ of finite PD presentation, then $A \rightarrow A'$ is also of finite PD presentation.*
- (ii) *If ρ is a PD saturated ideal of a PD ring R and $R \rightarrow R/\rho$ is of finite PD presentation, then there are finitely many elements $r_i \in R$ such that $\rho = \overline{(r)}$.*

Proof. (i) For any PD ring R and any element $r \in R$, we have $R[X]/\overline{(X-r)} = R[X]/(X-r) \cong R$, but we also have $R\langle \vec{X} \rangle/\overline{(X-r)} \cong R$ as long as r lies in the PD ideal of R . Now, let $A = R\langle \vec{X} \rangle[\vec{Y}]/\overline{(r)}$, $A' = R\langle \vec{X}' \rangle[\vec{Y}']/\overline{(r')}$, $f : A \rightarrow A'$, and choose x_i in the PD ideal of $R\langle \vec{X}' \rangle[\vec{Y}']$ and $y_i \in R\langle \vec{X}' \rangle[\vec{Y}']$ such that $f(X_i) = x_i$ and $f(Y_i) = y_i$ in A' . Then we can calculate:

$$\begin{aligned} A' &= A'\langle \vec{X} \rangle[\vec{Y}]/\overline{(X_i - x_i, Y_i - y_i)} \\ &= A'\langle \vec{X} \rangle[\vec{Y}]/\overline{(X_i - x_i, Y_i - y_i, r)} \\ &= A\langle \vec{X}' \rangle[\vec{Y}']/\overline{(X_i - x_i, Y_i - y_i, r')}. \end{aligned}$$

- (ii) Let $R\langle \vec{X} \rangle[\vec{Y}]/\overline{(r)} \cong R/\rho$. By choosing elements in I_R respectively R corresponding to the X_i and Y_i , this isomorphism is induced by a PD map $f : R\langle \vec{X} \rangle[\vec{Y}] \rightarrow R$. But then we have $\rho = \overline{(f(r_i))}$. \square

Lemma 5.6.5. *Homomorphisms of PD rings of finite type and homomorphisms of PD rings of finite presentation are both stable under composition.*

Proof. Let (K, I, γ) be a PD ring. We have

$$\left(K\langle \vec{X} \rangle[\vec{Y}]/\overline{(r)} \right) \langle \vec{X}' \rangle[\vec{Y}']/\overline{(r')} = K\langle \vec{X}, \vec{X}' \rangle[\vec{Y}, \vec{Y}']/\overline{(r, r')},$$

where r''_i is a lift of r'_i to $K\langle \vec{X}, \vec{X}' \rangle[\vec{Y}, \vec{Y}']$. Here, \vec{r} and \vec{r}' can either be finite lists or arbitrary set-indexed families. \square

Lemma 5.6.6. *Homomorphisms of PD rings of finite type and homomorphisms of PD rings of finite presentation are both stable under base change (pushout in the category of PD rings).*

Proof. The pushout of $K\langle \vec{X} \rangle[\vec{Y}]/\overline{(r)}$ along $f : (K, I, \gamma) \rightarrow (K', I', \gamma')$ is $K'\langle \vec{X} \rangle[\vec{Y}]/\overline{(f(r))}$. \square

Lemma 5.6.7. *Being of finite PD type and being of finite PD presentation are local properties of homomorphisms of PD rings, in the sense of [7, Tag 01SR], that is: Let $R = (R, I_R, \gamma_R)$ and $A = (A, I, \gamma)$ be PD rings.*

- (a) *If $R \rightarrow A$ is of finite PD type (presentation) and $r \in R$, then $R_r \rightarrow A_r$, with the induced PD structures, is of finite PD type (presentation).*
- (b) *If $r \in R$, $a \in A$ and $R_r \rightarrow A$ is of finite PD type (presentation), then $R \rightarrow A_a$ is of finite PD type (presentation).*
- (c) *If $R \rightarrow A$ is a homomorphism of PD rings, $a_i \in A$ are elements such that $(a_1, \dots, a_n) = (1)$, and $R \rightarrow A_{a_i}$ is of finite PD type (presentation) for every i , then $R \rightarrow A$ is of finite PD type (presentation).*

Proof. (a) This is a special case of Lemma 5.6.6.

(b) This follows from Lemma 5.6.5 and the fact that $R_r \cong R[X]/\overline{(rX-1)}$, and the same for A .

(c) Let the $R \rightarrow A_{a_i}$ be of finite PD type. For every i , choose elements $x_{i,j} a_i^{-m_{i,j}} \in I_{a_i}$ and $y_{i,j} a_i^{-n_{i,j}} \in A_{a_i}$ (with $x_{i,j} \in I$ and $y_{i,j} \in A$) according to Lemma 5.6.3. Then A_{a_i} is generated as an R -algebra by $\gamma_k(x_{i,j} a_i^{-m_{i,j}}) = \gamma_k(x_{i,j}) a_i^{-km_{i,j}}$ and $y_{i,j} a_i^{-n_{i,j}}$. Let $\sum_i c_i a_i = 1$ in A and set $\tilde{A} := R[a_i, c_i, \gamma_k(x_{i,j}), y_{i,j}] \subseteq A$. Then we see that for every element $a \in A$ and every i , we have $a_i^N a \in \tilde{A}$ for N big enough, which means that we also have $a = (\sum_i c_i a_i)^N a \in \tilde{A}$ for N big enough, so $\tilde{A} = A$. We also have (by Lemma 5.6.3) that I_{a_i} is generated as an ideal by the elements of I_R and the $\gamma_k(x_{i,j}) a_i^{-km_{i,j}}$. But then if $\tilde{I} \subseteq A$ is the ideal generated by I_R and all $\gamma(x_{i,j})$, we have $\tilde{I}_{a_i} = I_{a_i}$ for all i , which implies $\tilde{I} = I$. Thus, $R \rightarrow A$ is of finite PD type by the other direction of Lemma 5.6.3.

Now let the $R \rightarrow A_{a_i}$ be of finite PD presentation. We already know $A = R\langle \vec{X} \rangle[\vec{Y}]/\rho$ for a PD saturated ideal ρ . For every i , choose a representative $\tilde{a}_i \in R\langle \vec{X} \rangle[\vec{Y}]$ of $a_i \in A$. We note that $\rho_{\tilde{a}_i}$ is also saturated, and therefore $A_{a_i} = R\langle \vec{X} \rangle[\vec{Y}]_{\tilde{a}_i}/\rho_{\tilde{a}_i}$. Since both A_{a_i} and $R\langle \vec{X} \rangle[\vec{Y}]_{\tilde{a}_i}$ are of finite PD presentation over R , Lemma 5.6.4 tells us that $\rho_{\tilde{a}_i} = \overline{(r_{i,j} a_i^{-n_{i,j}})} = \overline{(r_{i,j})}$, with finitely many $r_{i,j} \in \rho$. Let $c_i \in R\langle \vec{X} \rangle[\vec{Y}]$ be elements with $1 - \sum_i c_i a_i \in \rho$. Then, setting $\tilde{\rho} := \overline{(1 - \sum_i c_i a_i, r_{i,j})} \subseteq R\langle \vec{X} \rangle[\vec{Y}]$, we have $\tilde{\rho} \subseteq \rho$, $(\tilde{a}_i) = (1)$ in $R\langle \vec{X} \rangle[\vec{Y}]/\tilde{\rho}$ and $\tilde{\rho}_{\tilde{a}_i} = \rho_{\tilde{a}_i}$ for all i . This implies $\tilde{\rho} = \rho$. \square

Definition 5.6.8. A morphism of PD schemes $(T, \mathcal{I}, \gamma) \rightarrow (S, \mathcal{I}_S, \gamma_S)$ is *locally of finite PD type (presentation)* if there are affine open coverings $T = \bigcup_i \text{Spec } A_i$ and $S = \bigcup_i \text{Spec } K_j$ such that every $\text{Spec } A_i$ maps to some $\text{Spec } K_{j_i}$ where $K_{j_i} \rightarrow A_i$ (with the induced structures of PD rings) is of finite PD type (presentation).

The following good properties of morphisms locally of finite PD type and morphisms locally of finite PD presentation now follow from Lemmas 5.6.5, 5.6.6 and 5.6.7 as in

[7, Tag 01SQ]: If $T \rightarrow S$ is locally of finite PD type (presentation), then for any affine open $\text{Spec } A \subseteq T$ mapping into an affine open $\text{Spec } K \subseteq S$, the homomorphism of PD rings $K \rightarrow A$ is of finite PD type (presentation). In particular, the morphisms locally of finite PD type (presentation) between affine PD schemes correspond to the morphisms of finite PD type (presentation) between PD rings. And morphisms locally of finite PD type (presentation) are stable under composition and base change.

Definition 5.6.9. We say that the PD ideal I of a PD ring (A, I, γ) is *PD-generated* by a subset $G \subseteq I$ if $\overline{(G)} = I$, that is, if it is generated (as an ideal) by the elements $\gamma_n(g)$ for $g \in G$ and $n \geq 1$. We say I is *finitely PD-generated* if it is PD-generated by a finite set G .

We also globalize the notion of a PD ideal being finitely PD-generated. Note that by Lemma 5.6.4, if (A, I, γ) is a PD ring and we equip A/I with the trivial PD structure, then I is finitely PD-generated if and only if $A \rightarrow A/I$ is of finite PD presentation.

Definition 5.6.10. The PD ideal sheaf \mathcal{I} of a PD scheme (S, \mathcal{I}, γ) is *locally finitely PD-generated* if for every affine open $U \subseteq S$, the PD ideal $\mathcal{I}(U) \subseteq \mathcal{O}_S(U)$ is finitely PD-generated, that is, if the closed embedding $V(\mathcal{I}) \rightarrow S$ is locally of finite PD presentation, where $V(\mathcal{I})$ is regarded as a PD scheme with trivial PD structure.

5.7 Syntactic presentation of the big crystalline topos

We can now finish the definition of the precise variant of the crystalline topos for which we will give a syntactic presentation. The objects we want to allow in the site are those PD thickenings (T, \mathcal{I}, γ) over S and X where $T \rightarrow S$ is locally of finite PD presentation. An essentially small dense subcategory is for example the one where T is additionally required to be an open subscheme of an affine scheme which is mapped into an affine open of S . We also impose a finiteness assumption on X without which the connection to compact models motivating this definition would be lost.

Definition 5.7.1. Let $(S, \mathcal{I}_S, \gamma_S)$ be a PD scheme and let X be a scheme locally of finite presentation over $S_0 = V(\mathcal{I}_S)$. Then we set

$$(X/S)_{\text{Cris}_{\text{fp}}} := (X/S)_{\text{Cris}_{\mathcal{C}}},$$

where \mathcal{C} is the class of PD thickenings (T, \mathcal{I}, γ) over S and X for which $T \rightarrow S$ is locally of finite PD presentation.

Now there is not much missing for the proof of the classification result. The following lemma will be used to show that the axiom (nil) is a rigid-topology quotient.

Lemma 5.7.2. *Let $(A, \mathfrak{a}, \gamma)$ be a PD ring. If \mathfrak{a} is PD-generated by nilpotent elements, then \mathfrak{a} is a nil ideal.*

Proof. We need to show that \mathfrak{a} is generated (as an ordinary ideal) by nilpotent elements. So if $a \in \mathfrak{a}$ is nilpotent, say $a^e = 0$, we need to show that $\gamma_n(a)$ is still nilpotent for

every $n \geq 1$. It is most convenient to do the necessary calculation in $\mathbb{Q}[X]$ first (with the unique PD structure γ on any ideal):

$$(\gamma_n(X))^k = \frac{1}{(n!)^k} X^{kn} = \frac{(kn-e)!}{(n!)^k} X^e \gamma_{kn-e}(X).$$

One can check that $c := \frac{(kn-e)!}{(n!)^k}$ is an integer for k big enough ($k \geq \lceil \frac{e}{n} \rceil (n+1)$ suffices). But then we have the equation $(\gamma_n(X))^k = cX^e \gamma_{kn-e}(X)$ also in the sub-PD-ring $\mathbb{Z}\langle X \rangle \subseteq \mathbb{Q}[X]$, and then, via the unique PD morphism $\mathbb{Z}\langle X \rangle \rightarrow A$ sending X to a , we obtain $(\gamma_n(a))^k = 0$ for sufficiently large k . \square

Proposition 5.7.3. *Let (K, I_K, γ_K) be a PD ring with I_K finitely PD-generated and let R be a finitely presented K/I_K -algebra. Then the theory $K\text{-Alg-}R\text{-Quot} + \text{PD}_{\gamma_K} + (\text{nil})$ is of presheaf type and its compact models are those $(\gamma \curvearrowright I \triangleleft A \twoheadrightarrow B)$ where (A, I, γ) is of finite PD presentation over (K, I_K, γ_K) .*

Proof. We start with the cartesian theory $K\text{-Alg} + \text{Ideal}_{I_K} + \text{PD}_{\gamma_K}$. By Lemma 5.6.1, the compact models are the PD rings of finite PD presentation over (K, I_K, γ_K) . It is an equivalence extension to add a sort B with K -algebra structure for the quotient A/I , arriving at $K\text{-Alg-}(K/I_K)\text{-Quot} + \text{PD}_{\gamma_K}$. Since R is a finitely presented K/I_K -algebra, finitely many constant symbols and finitely many axioms in the empty context suffice to add an R -algebra structure to B , so by Proposition 4.4.5 and Corollary 4.4.2, $K\text{-Alg-}R\text{-Quot} + \text{PD}_{\gamma_K}$ is of presheaf type and a model $(\gamma \curvearrowright I \triangleleft A \twoheadrightarrow B)$ is still compact if and only if (A, I, γ) is of finite PD presentation over K . All that remains is to show that (nil) is a rigid-topology quotient of this theory.

Let a compact model $M = (\gamma \curvearrowright I \triangleleft A \twoheadrightarrow B) \in (K\text{-Alg-}R\text{-Quot} + \text{PD}_{\gamma_K})\text{-mod}(\text{Set})_c$ be given. Since I_K is finitely PD-generated and A is of finite PD presentation over K , the PD ideal I is finitely PD-generated. Let $a \in I$ be one of the PD generators in a chosen finite family. The $J_{(\text{nil})}$ -covering cosieve S_a of Theorem 4.3.1 is the cosieve of all arrows $M \rightarrow M'$ that send a to an nilpotent element; it is generated by the countable family of the arrows $M \rightarrow M^n = (\gamma \curvearrowright I/\overline{(a^n)} \triangleleft A/\overline{(a^n)} \twoheadrightarrow B)$. In each of the (still compact) models M^n , the PD ideal $I/\overline{(a^n)}$ is PD generated by the images of the PD generators of I , so we can apply this construction to the other PD generators in turn to cover M by models where the PD ideal is generated by nilpotent elements and where thus, by Lemma 5.7.2, the axiom (nil) is fulfilled. \square

Theorem 5.7.4. *Let (K, I_K, γ_K) be a PD ring with I_K finitely PD-generated and let R be a finitely presented K/I_K -algebra. Then the big crystalline topos*

$$(\text{Spec } R/\text{Spec } K)_{\text{Cris}_{\text{fp}}}$$

classifies the geometric theory

$$K\text{-Alg-}R\text{-Quot} + \text{PD}_{\gamma_K} + (\text{nil}) + (\text{loc}).$$

The universal model is the short exact sequence associated to the structure sheaf

$$\mathcal{J} \hookrightarrow \mathcal{O} \twoheadrightarrow \mathcal{O}',$$

with the canonical K -algebra structure on \mathcal{O} and R -algebra structure on \mathcal{O}' .

Proof. Set

$$\mathbb{T}_0 := K\text{-Alg-}R\text{-Quot} + \text{PD}_{\gamma_K} + (\text{nil}).$$

The underlying category of the site of affine objects for $(\text{Spec } R / \text{Spec } K)_{\text{Cris}_{\text{fp}}}$ is dual to the subcategory of $\mathbb{T}_0\text{-mod}(\text{Set})$ where (A, I, γ) is of finite PD presentation over K . By Proposition 5.7.3, this is exactly $\mathbb{T}_0\text{-mod}(\text{Set})_c$, and the associated presheaf topos

$$[\mathbb{T}_0\text{-mod}(\text{Set})_c, \text{Set}]$$

classifies \mathbb{T}_0 . The Zariski topology on this site, that is, the restriction of the one in Lemma 5.4.5 to our compact models, coincides with the topology induced by (nil) via Theorem 4.3.1 (just as in Corollary 5.5.5).

For the universal model, we note that the structure sheaf model $\mathcal{J} \hookrightarrow \mathcal{O} \twoheadrightarrow \mathcal{O}'$, when restricted to the site of affine objects, becomes simply the tautological model in $[\mathbb{T}_0\text{-mod}(\text{Set})_c, \text{Set}]$, which is the universal model of \mathbb{T}_0 . In particular, the presheaves of which the universal \mathbb{T}_0 -model consists are sheaves for the topology induced by (loc) , implying that the same model, regarded as a model in the subtopos, is also the universal model of $\mathbb{T}_0 + (\text{loc})$. \square

Corollary 5.7.5. *Let K, R be as in Theorem 5.7.4. Then the category of points of the big crystalline topos $(\text{Spec } R / \text{Spec } K)_{\text{Cris}_{\text{fp}}}$ is equivalent to*

$$\left(K\text{-Alg-}R\text{-Quot} + \text{PD}_{\gamma_K} + (\text{nil}) + (\text{loc}) \right)\text{-mod}(\text{Set}),$$

the category of affine PD thickenings (A, I, γ) over (K, I_K, γ_K) (implying that I is a nil ideal) together with an R -algebra structure on A/I , such that A is a local ring.

Proof. The points of a topos are by definition the geometric morphisms from Set to it. So this follows immediately from Theorem 5.7.4 and the definition of classifying topos. \square

Remark 5.7.6. If the ring R has non-zero characteristic, then there is a different way of dealing with the axiom (nil) . Namely, one can see that if the PD ideal I of A contains a non-zero integer, then (nil) is actually equivalent to

$$\top \vdash_{\square} \bigvee_{0 \neq n \in \mathbb{Z}} (c_n = 0),$$

which is an axiom in the empty context. So in Theorem 5.7.4, instead of assuming that I_K is finitely PD-generated, we can alternatively assume R to have non-zero characteristic.

5.8 The non-affine case

Now we apply the results of Section 3 to the big crystalline topos to obtain a syntactic presentation of $(X/S)_{\text{Cris}_{\text{fp}}}$ in the case where S and X are not affine. This will be an adaptation of the treatment of the Zariski topos in Section 3 to the crystalline topos.

and we will mostly be concerned with the aspects which differ in these two situations and how to handle them.

Since we only know a theory classified by $(X/S)_{\text{Cris}_{\text{fp}}}$ in the case where both S and X are affine, we will want to cover both S and X by affine opens S_i respectively X_i such that for each i , X_i lies over S_i .

$$\begin{array}{ccc} S & \longleftarrow & X \\ \uparrow & & \uparrow \\ S_i & \longleftarrow & X_i \end{array}$$

This is always possible, as we can first choose an affine cover of S by some S_i and then cover the $X \times_S S_i$ by affine opens. Also, an open subscheme $S' \subseteq S$ is canonically a PD scheme as $(S', \mathcal{I}_S|_{S'}, \gamma_S|_{S'})$, and we note that if $X' \rightarrow S$ factors through S' , then it automatically also factors through $S'_0 = V(\mathcal{I}_S|_{S'})$.

When we turn to defining open subtoposes corresponding to such pairs of open subschemes (S', X') , there are two things to note. Firstly, it is no longer the case that (S', X') is an object of the site $\text{Cris}_{\text{fp}}(X/S)$, since X' is not at all required to be isomorphic to S'_0 , and the PD scheme $(S', \mathcal{I}_S|_{S'}, \gamma_S|_{S'})$ itself does not have to be a PD thickening. But we can still consider the class of all objects $(T, \mathcal{I}, \gamma) \in \text{Cris}_{\text{fp}}(X/S)$ such that $T \rightarrow S$ factors through S' and $V(\mathcal{I}) \rightarrow X$ factors through X' , that is, the subterminal presheaf

$$U_{(S', X')} := \text{Hom}_{X/S}(-, (S', X')) : \text{Cris}_{\text{fp}}(X/S)^{\text{op}} \rightarrow \text{Set},$$

and it is easy to see that this is in fact a sheaf for the Zariski topology on $\text{Cris}_{\text{fp}}(X/S)$.

Secondly, however, this subterminal object $U_{(S', X')}$ depends in fact only on X' and not on S' , namely, we have $U_{(S', X')} = U_{(S, X')}$. This is because the closed embedding $V(\mathcal{I}) \hookrightarrow T$ is a homeomorphism, and whether or not $T \rightarrow S$ factors through the open subscheme S' can be tested on the level of points.

$$\begin{array}{ccc} S & \longleftarrow & X \\ \uparrow & & \uparrow \\ S' & \longleftarrow & X' \\ \uparrow & & \uparrow \\ T & \longleftarrow & V(\mathcal{I}) \end{array}$$

(A curved arrow on the left points from T to S)

Thus, we only have induced open subtoposes $U_{X'} := U_{(S, X')}$ for all open subschemes $X' \subseteq X$.

This leads to the slightly subtle situation that a cover of X by affine open X_i with the property that every $X_i \rightarrow S$ factors through some affine open $S_i \subseteq S$ induces an open cover of $(X/S)_{\text{Cris}_{\text{fp}}}$ by subtoposes for which we know classified theories, but if $S_i = \text{Spec } K_i$ and $S'_i = \text{Spec } K'_i$ are two different choices for the same $X_i = \text{Spec } R_i$, then we have no direct algebraic relation between K_i and K'_i , so we have no way to construct a diagonal extension between the classified theories. Therefore it seems more reasonable

to work with pairs (S_i, X_i) of open subschemes from the beginning and only drop the condition that the S_i cover S .

Here is the analogue of Lemma 3.7.3 that we need.

Lemma 5.8.1.

(i) *The mapping*

$$X' \mapsto ((X/S)_{\text{Cris}_{\text{fp}}})_{o(U_{X'})}$$

from open subschemes of X to open subtoposes of $(X/S)_{\text{Zar}_{\text{fp}}}$ is monotone and preserves finite intersections and arbitrary unions.

(ii) *For open subschemes $S' \subseteq S$ and $X' \subseteq X$ with $X' \rightarrow S'$, the open subtopos $((X/S)_{\text{Cris}_{\text{fp}}})_{o(U_{X'})}$ is equivalent to $(X'/S')_{\text{Cris}_{\text{fp}}}$.*

(iii) *If $S = \text{Spec } K$ and $X = \text{Spec } R$ are affine, with I_K finitely PD-generated and R finitely presented over K/I_K , so that $(X/S)_{\text{Cris}_{\text{fp}}}$ classifies*

$$\mathbb{T}_{K,R} := K\text{-Alg-}R\text{-Quot} + \text{PD}_{\gamma_K} + (\text{nil}) + (\text{loc}),$$

and $h \in R$, then the open subtopos $((X/S)_{\text{Cris}_{\text{fp}}})_{o(U_{D(h)})}$ is presented by the closed geometric formula

$$\text{inv}(c_h),$$

which is also equivalent to

$$\text{inv}(c_g : A) \wedge \text{inv}(c_h : B)$$

for any $g \in K$ such that $g \mid h$ in R .

Proof. (i) Monotonicity and finite intersections are clear. For unions, let X'_i be a family of open subschemes of X and let $F \subseteq 1_{(X/S)_{\text{Cris}_{\text{fp}}}}$ be a subterminal sheaf with $U_{X'_i} \leq F$ for all i . We want to show $U_{X'} \leq F$ for $X' := \bigcup_i X'_i$. So let $T = (T, \mathcal{I}, \gamma) \in \text{Cris}_{\text{fp}}(X/S)$ be given with $V(\mathcal{I}) \rightarrow X'$. By pulling back the X'_i , this induced an open cover of $V(\mathcal{I})$, or equivalently an open cover $T = \bigcup_i T_i$, such that $V(\mathcal{I}|_{T_i}) \rightarrow X_i$. This means $|U_{X_i}(T_i)| = 1$, which implies $|F(T_i)| = 1$, and since F is a sheaf for the Zariski topology on $\text{Cris}_{\text{fp}}(X/S)$, we obtain $|F(T)| = 1$.

(ii) As explained above, $U_{X'}$ selects exactly those objects of $\text{Cris}_{\text{fp}}(X/S)$ which belong to $\text{Cris}_{\text{fp}}(X'/S')$. The rest is exactly as in the case of the Zariski topos.

(iii) We know that $(X/S)_{\text{Cris}_{\text{fp}}}$ classifies $\mathbb{T}_{K,R}$ and

$$((X/S)_{\text{Cris}_{\text{fp}}})_{o(U_{D(h)})} \simeq (D(h)/S)_{\text{Cris}_{\text{fp}}}$$

classifies \mathbb{T}_{K,R_h} , which is equivalent to $\mathbb{T}_{K,R} + \text{inv}(c_h)$. It follows that $\text{inv}(c_h)$ presents this open subtopos, since one structure sheaf pulls back to the other,

including the extra structure that makes up the universal models. For $g \in K$ with $g \mid h$ in R , the theory $\mathbb{T}_{K,R}$ shows

$$\text{inv}(c_h) \vdash_{\square} \text{inv}(f(c_g))$$

and also, by Remark 5.5.6,

$$\text{inv}(f(c_g)) \dashv\vdash_{\square} \text{inv}(c_g).$$

□

As the final prerequisite, we show that intersections of pairs of open subschemes can be covered by appropriate pairs of standard opens.

Lemma 5.8.2. *Let $X \rightarrow S$ be a morphism of schemes, let $S_1, S_2 \subseteq S$ and let $X_1, X_2 \subseteq X$ be affine open subschemes, $S_i = \text{Spec } K_i$, $X_i = \text{Spec } R_i$, such that $X_i \rightarrow S_i$. Then there are families of elements $g_i^j \in K_i$, $h_i^j \in R_i$, $j \in J$, $i \in \{1, 2\}$, such that for every $j \in J$ we have $g_i^j \mid h_i^j$ in R_i and*

$$\text{Spec } K_1 \supseteq D(g_1^j) = D(g_2^j) \subseteq \text{Spec } K_2,$$

$$\text{Spec } R_1 \supseteq D(h_1^j) = D(h_2^j) \subseteq \text{Spec } R_2$$

as open subschemes of S respectively X , and

$$X_1 \cap X_2 = \bigcup_{j \in J} D(g_1^j) = \bigcup_{j \in J} D(g_2^j).$$

Proof. As mentioned before, we can cover $S_1 \cap S_2$ by open subschemes simultaneously standard open in S_1 and S_2 . The same is true for $X_1 \cap X_2$, and since the simultaneously standard opens even form a basis of opens of $X_1 \cap X_2$, we can also arrange that $D(h_1^j)$ maps to $D(g_1^j)$, and, equivalently, $D(h_2^j)$ maps to $D(g_2^j)$. This means that for each i and j , we have the dashed arrow in

$$\begin{array}{ccc} R_i & \longrightarrow & (R_i)_{h_i^j} \\ \uparrow & & \uparrow \text{---} \\ K_i & \longrightarrow & (K_i)_{g_i^j}, \end{array}$$

so g_i^j is invertible in $(R_i)_{h_i^j}$, so g_i^j divides some power of h_i^j , and replacing h_i^j with this power, we obtain $g_i^j \mid h_i^j$ in R_i . □

Recall the notation $R\text{-AlgStr}_K(B)$ for the extension adding to a sort B an R -algebra structure compatible with a previously given K -algebra structure, from Definition 5.3.2. For a PD ring (K, I_K, γ_K) , we also denote γ/γ_K the axioms distinguishing the extension PD_{γ_K} of $K\text{-Alg} + \text{Ideal}_{I_K}$ from PD.

Theorem 5.8.3. *Let $(S, \mathcal{I}_S, \gamma_S)$ be a PD scheme with \mathcal{I}_S locally finitely PD-generated, and let X be a scheme locally of finite presentation over $S_0 = V(\mathcal{I}_S)$. Let*

$$\mathrm{Spec} K_i = S_i \subseteq S, \quad \mathrm{Spec} R_i = X_i \subseteq X$$

be open subschemes such that $X_i \rightarrow S_i$ and $X = \bigcup_{i \in I} X_i$. For every $i \neq i' \in I$, let

$$g_{i,i'}^j \in K_i, \quad g_{i',i}^j \in K_{i'}, \quad h_{i,i'}^j \in R_i, \quad h_{i',i}^j \in R_{i'},$$

$j \in J_{\{i,i'\}}$, be families of elements as in Lemma 5.8.2, with corresponding ring isomorphisms

$$\varphi_{i,i'}^j = (\varphi_{i',i}^j)^{-1} : (K_i)_{g_{i,i'}^j} \rightarrow (K_{i'})_{g_{i',i}^j}, \quad \tilde{\varphi}_{i,i'}^j = (\tilde{\varphi}_{i',i}^j)^{-1} : (R_i)_{h_{i,i'}^j} \rightarrow (R_{i'})_{h_{i',i}^j}.$$

Then $(X/S)_{\mathrm{Crisfp}}$ classifies the geometric theory

$$\begin{aligned} \mathbb{T}_{X/S} := & \mathbb{Z}\text{-Alg-}\mathbb{Z}\text{-Quot} + \mathrm{PD} + (\mathrm{nil}) + (\mathrm{loc}) + \langle p_i \rangle_{i \in I} + \left(\bigvee_{i \in I} p_i \right) \\ & + \left((K_i\text{-AlgStr}(A) + R_i\text{-AlgStr}_{K_i}(B) + \gamma/\gamma_{K_i}) / p_i \right)_{i \in I} \\ & + \left(x \in \tilde{c}_{h_{i,i'}^j} \wedge \mathrm{inv}(x) \vdash_{x:B} p_{i'} \right)_{i \neq i' \in I, j \in J_{\{i,i'\}}} \\ & + \left(p_i \wedge p_{i'} \vdash_{\square} \bigvee_{j \in J_{\{i,i'\}}} \exists x : B. (x \in \tilde{c}_{h_{i,i'}^j} \wedge \mathrm{inv}(x)) \right)_{i \neq i' \in I, j \in J_{\{i,i'\}}} \\ & + \left(x \in \tilde{c}_{g_{i',i}^j} \wedge \mathrm{inv}(x) \wedge y \in \tilde{c}_\lambda \wedge z \in \tilde{c}_{\lambda'} \vdash_{x,y,z:A} x^n y = z \right)_{i \neq i' \in I, j \in J_{\{i,i'\}}, \lambda \in K_i} \\ & + \left(x \in \tilde{c}_{h_{i',i}^j} \wedge \mathrm{inv}(x) \wedge y \in \tilde{c}_\mu \wedge z \in \tilde{c}_{\mu'} \vdash_{x,y,z:B} x^n y = z \right)_{i \neq i' \in I, j \in J_{\{i,i'\}}, \mu \in R_{i'}} \end{aligned}$$

where in the last two families of axioms, $\lambda' \in K_{i'}$ and $n \in \mathbb{N}$ are chosen for each $\lambda \in K_i$ such that $\varphi_{i,i'}^j(\lambda) = (g_{i',i}^j)^{-n} \lambda'$, and $\mu' \in R_{i'}$ and $n \in \mathbb{N}$ are chosen for each $\mu \in R_i$ such that $\tilde{\varphi}_{i,i'}^j(\mu) = (h_{i',i}^j)^{-n} \mu'$,

Proof. Since \mathcal{I}_S is locally finitely PD-generated, the PD ideals $I_{K_i} := \mathcal{I}_S(S_i) \subseteq K_i$ with PD structure $\gamma_{K_i} := \gamma_S(S_i)$ are finitely PD-generated, and since X is locally of finite presentation over S_0 , each R_i is a finitely presented K_i/I_{K_i} -algebra. So $(X_i/S_i)_{\mathrm{Crisfp}}$ classifies

$$\mathbb{T}_{K_i, R_i} = K_i\text{-Alg-}R_i\text{-Quot} + \mathrm{PD}_{\gamma_{K_i}} + (\mathrm{nil}) + (\mathrm{loc}).$$

By Lemma 5.8.1, these toposes

$$\mathcal{E}_i := ((X/S)_{\mathrm{Crisfp}})_{o(U_{X_i})} \simeq (X_i/S_i)_{\mathrm{Crisfp}}$$

form an open cover of $(X/S)_{\mathrm{Crisfp}}$, and the universal models are extensions of the restrictions of the short exact sequence $\mathcal{J} \hookrightarrow \mathcal{O} \twoheadrightarrow \mathcal{O}'$ surrounding the structure sheaf. So we can use the base theory and base model

$$\begin{aligned} \mathbb{T}_0 &:= \mathbb{Z}\text{-Alg-}\mathbb{Z}\text{-Quot} + \mathrm{PD} + (\mathrm{nil}) + (\mathrm{loc}), \\ M_0 &:= (\mathcal{J} \hookrightarrow \mathcal{O} \twoheadrightarrow \mathcal{O}'), \end{aligned}$$

where \mathbb{T}_0 is of course equivalent to $\text{Ring} + \text{Ideal} + \text{PD} + (\text{nil}) + (\text{loc})$, but we want to have the sort $B = A/I$ in the base theory. And we have presentations of the \mathcal{E}_i over (\mathbb{T}_0, M_0) with

$$\mathbb{E}_i := K_i\text{-AlgStr}(A) + R_i\text{-AlgStr}_{K_i}(B) + \gamma/\gamma_{K_i}.$$

Closed geometric formulas for the intersections $\mathcal{E}_i \cap \mathcal{E}_{i'} \subseteq \mathcal{E}_i$ are given by

$$\phi_{i,i'} := \bigvee_{j \in J_{\{i,i'\}}} \text{inv}(c_{h_{i,i'}^j}).$$

The diagonal quotient extensions $\mathbb{Q}_{\{i,i'\}}$ have to make both the $K_{i'}$ -algebra structure on A and the $R_{i'}$ -algebra structure on B definable in terms of $\mathbb{T}_0 + \mathbb{E}_i + \phi_{i,i'}$. Since the formula $\phi_{i,i'}$ also implies

$$\bigvee_{j \in J_{\{i,i'\}}} \text{inv}(c_{g_{i,i'}^j}),$$

we can proceed as in Theorem 3.7.6 for both K_i and R_i . That is, for every $\lambda \in K_i$ and $\mu \in R_i$, we write

$$\varphi_{i,i'}^j(\lambda) = (g_{i',i}^j)^{-n} \lambda'$$

for some $\lambda' \in K_{i'}$, $n \in \mathbb{N}$, respectively

$$\tilde{\varphi}_{i,i'}^j(\mu) = (h_{i',i}^j)^{-n} \mu'$$

for some $\mu' \in R_{i'}$, $n \in \mathbb{N}$. Then $\mathcal{O}_{((X_i \cap X_{i'})/(S_i \cap S_{i'}))_{\text{Cris}_{\text{fp}}}}$ satisfies

$$\begin{aligned} \mathbb{Q}_{i,i'} &:= \left\{ \text{inv}(c_{g_{i',i}^j}) \vdash_{\square} (c_{g_{i',i}^j})^n c_{\lambda} = c_{\lambda'} \mid j \in J_{\{i,i'\}}, \lambda \in K_i \right\}, \\ \tilde{\mathbb{Q}}_{i,i'} &:= \left\{ \text{inv}(c_{h_{i',i}^j}) \vdash_{\square} (c_{h_{i',i}^j})^n c_{\mu} = c_{\mu'} \mid j \in J_{\{i,i'\}}, \mu \in R_i \right\}, \end{aligned}$$

and

$$\mathbb{Q}_{\{i,i'\}} := \mathbb{Q}_{i,i'} + \mathbb{Q}_{i',i} + \tilde{\mathbb{Q}}_{i,i'} + \tilde{\mathbb{Q}}_{i',i}$$

is a diagonal quotient extension of $\mathbb{E}_i + \phi_{i,i'}$ and $\mathbb{E}_{i'} + \phi_{i',i}$ over \mathbb{T}_0 .

Applying Corollary 3.6.1 to these data and simplifying the axioms slightly yields the theory $\mathbb{T}_{X/S}$ in the statement. \square

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